Transformations and Transitions from the Sylvester to the Bezout Resultant

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Abstract

A simple matrix transformation linking the resultant matrices of Sylvester and Bezout is derived. This transformation matrix is then applied to generate an explicit formula for each entry of the Bezout resultant, and this entry formula is used, in turn, to construct an efficient recursive algorithm for computing all the entries of the Bezout matrix. Hybrid resultant matrices consisting of some columns from the Sylvester matrix and some columns from the Bezout matrix provide natural transitions from the Sylvester to the Bezout resultant, and allow as well the Bezout construction to be generalized to two polynomials of different degrees. Such hybrid resultants are derived here, employing again the transformation matrix from the Sylvester to the Bezout resultant.

1 Introduction

Resultants play an important role in algorithmic algebraic geometry [Cox *et al* 1997], computer aided geometric design [Goldman *et al* 1984; De Montaudouin and Tiller 1984; Sederberg and Parry 1986; Manocha and Canny 1992], and robotics [Canny 1987; Canny 1988]. For two univariate polynomials of the same degree, there are two standard matrix representations for the resultant: the Sylvester resultant [van der Waerden 1950] and the Bezout resultant [Goldman *et al* 1984; De Montaudouin and Tiller 1984]. Typically these two resultants are constructed in isolation, without any regard for how one matrix is related to the other. The purpose of this paper is to explore a variety of mathematical connections between the resultant matrices of Sylvester and Bezout.

We shall derive a simple, block symmetric, block upper triangular matrix that transforms the Sylvester matrix into the Bezout matrix. This matrix transformation captures the essence of the mathematical relationships between these two resultant matrices. We shall then apply this transformation matrix to:

1. derive an explicit formula for each entry of the Bezout matrix;
2. develop an efficient recursive algorithm for computing all the entries of the Bezout matrix;

3. construct a sequence of hybrid resultant matrices that provide a natural transition from the Sylvester matrix to the Bezout matrix, each hybrid consisting of some columns from the Sylvester matrix and some columns from the Bezout matrix;

4. extend the Bezout construction to two univariate polynomials of different degrees.

Except for the recursive algorithm, all of these results are already known [Goldman *et al* 1984; Sederberg *et al* 1997; Cox *et al* 1997]. What is new here in addition to the recursive algorithm is our approach: deriving all these properties in a unified fashion based on the structure of the transformation matrix. Equally important, these seemingly straightforward features of the resultants for two univariate polynomials of degree n anticipate similar, but much more intricate, properties and interrelationships between the resultants for three bivariate polynomials of bidegree (m, n) [Chionh *et al* 1998; Zhang *et al* 1998].

This paper is organized in the following fashion. In Section 2 we briefly review the standard constructions for the Sylvester and Bezout resultants. In Section 3 we introduce the method of truncated formal power series, and in Section 4 we apply this method to generate the matrix transformation from the Sylvester matrix to the Bezout matrix. Section 5 is devoted to developing a new, efficient, recursive algorithm for computing all the entries of the Bezout resultant. Hybrid resultants are discussed in Section 6. We close in Section 7 with a derivation of the Bezout resultant for two univariate polynomials of different degree.

2 The Sylvester and Bezout Resultants

Let $f(t) = \sum_{i=0}^{n} a_i t^i$, $g(t) = \sum_{j=0}^{n} b_j t^j$ be two polynomials of degree n, and let $L(t) = [\ f(t) \ \ g(t)\ ]$. The determinant of the $2n \times 2n$ coefficient matrix of the $2n$ polynomials $t^\tau L$, $0 \le \tau \le n-1$, is known as the Sylvester resultant of f and g. To fix the order of the rows and columns of the Sylvester matrix S_n, we define

$$\begin{bmatrix} L & tL & \cdots & t^{n-1}L \end{bmatrix} = \begin{bmatrix} 1 & \cdots & t^{2n-1} \end{bmatrix} S_n. \tag{1}$$

Let $L_i = [\ a_i \ \ b_i\ ]$; then

$$S_n = \begin{bmatrix} L_0 & & \\ \vdots & \ddots & \\ L_{n-1} & \ddots & L_0 \\ L_n & \ddots & L_1 \\ & \ddots & \vdots \\ & & L_n \end{bmatrix}.$$

Note that we have adopted a somewhat nonstandard form for the Sylvester matrix by grouping together the polynomials f and g (compare to [van der Waerden 1950]). This grouping will simplify our work later on and will change at most the sign of the Sylvester determinant. Observe too that with this ordering of the columns the top (bottom) half of the Sylvester matrix is striped, block lower (upper) triangular, and block symmetric with respect to the northeast-southwest diagonal.

To obtain the Bezout matrix for f and g, we consider the Cayley expression

$$\Delta_n(t, \beta) = \begin{vmatrix} f(t) & g(t) \\ f(\beta) & g(\beta) \end{vmatrix} / (\beta - t). \tag{2}$$

Since the denominator exactly divides the numerator in $\Delta_n(t, \beta)$, this rational Cayley expression is really a degree $n-1$ polynomial in t and in β, so we can write

$$\Delta_n(t, \beta) = \sum_{v=0}^{n-1} D_v(t) \beta^v \tag{3}$$

where $D_v(t)$, $0 \le v \le n-1$, are polynomials of degree $n-1$ in t. The determinant of the $n \times n$ coefficient matrix of the n polynomials $D_0(t), \cdots, D_{n-1}(t)$ is known as the Bezout resultant of f and g.

To fix the order of the rows and columns of the Bezout matrix B_n, we define

$$\sum_{v=0}^{n-1} D_v(t)\beta^v = \left[\begin{array}{ccc} 1 & \cdots & t^{n-1} \end{array}\right] B_n \left[\begin{array}{c} 1 \\ \vdots \\ \beta^{n-1} \end{array}\right]. \tag{4}$$

Thus the rows of B_n are indexed by $1, t, \cdots, t^{n-1}$ and the columns by $1, \beta, \cdots, \beta^{n-1}$.

It is well-known that the Bezout matrix B_n is symmetric [Goldman *et al* 1984]. Here we provide an elementary, high level, proof of this fact without computing entry formulas for B_n. Since

$$\Delta_n(t,\beta) = \left|\begin{array}{cc} f(t) & g(t) \\ f(\beta) & g(\beta) \end{array}\right| /(\beta - t) = \left|\begin{array}{cc} f(\beta) & g(\beta) \\ f(t) & g(t) \end{array}\right| /(t - \beta) = \Delta_n(\beta, t), \tag{5}$$

we have

$$\begin{aligned}
[1 \ \cdots \ \beta^{n-1}]\ B_n^T \left[\begin{array}{c} 1 \\ \vdots \\ t^{n-1} \end{array}\right] &= \left([1 \ \cdots \ \beta^{n-1}]\ B_n^T \left[\begin{array}{c} 1 \\ \vdots \\ t^{n-1} \end{array}\right]\right)^T \\
&= [1 \ \cdots \ t^{n-1}]\, B_n \left[\begin{array}{c} 1 \\ \vdots \\ \beta^{n-1} \end{array}\right] \\
&= \Delta_n(t,\beta) \\
&= \Delta_n(\beta,t) \qquad \text{[by Equation (5)]} \\
&= [1 \ \cdots \ \beta^{n-1}]\, B_n \left[\begin{array}{c} 1 \\ \vdots \\ t^{n-1} \end{array}\right].
\end{aligned}$$

Hence $B_n^T = B_n$, so B_n is symmetric.

The columns of the Sylvester matrix S_n represent polynomials of degree $2n-1$, whereas the columns of the Bezout matrix B_n represent polynomials of degree $n-1$. Since, in general, $|S_n| \neq 0$, the columns of S_n are linearly independent. Hence the $2n$ polynomials represented by these columns span the space of polynomials of degree $2n-1$. In particular, we can express the polynomials represented by the columns of the Bezout matrix as linear combinations of the polynomials represented by the columns of the Sylvester matrix. Thus there must be a matrix that transforms the Sylvester matrix into the Bezout matrix. In the next section we introduce a mathematical technique that will help us to find explicit formulas for the entries of this transformation matrix.

3 Exact Division by Truncated Formal Power Series

Let $\mu(x,y) = \sum_{i=0}^{m} a_i(y)x^i$ be a polynomial in x and y. If the rational expression $\mu(x,y)/(x-y)$ is actually a polynomial— that is, if $x-y$ exactly divides $\mu(x,y)$— we can convert the division to multiplication by replacing each monomial x^i in the numerator by the sum $\sum_{k=0}^{i-1} y^{i-1-k}x^k$. That is,

$$\sum_{i=0}^{m} a_i(y)x^i/(x-y) = \sum_{i=0}^{m} a_i(y) \sum_{k=0}^{i-1} y^{i-1-k}x^k, \tag{6}$$

where the vacuous sum $\sum_{k=0}^{-1}$ is taken to be zero. This identity is motivated by the observation that formally we have

$$\frac{1}{x-y} = \sum_{k=0}^{\infty} \frac{y^k}{x^{k+1}}.$$

Thus

$$
\begin{aligned}
x^i \sum_{k=0}^{\infty} \frac{y^k}{x^{k+1}} &= x^i \sum_{k=0}^{i-1} \frac{y^k}{x^{k+1}} + x^i \sum_{k=i}^{\infty} \frac{y^k}{x^{k+1}} \\
&= \sum_{k=0}^{i-1} y^{i-1-k} x^k + \text{terms involving negative powers of } x.
\end{aligned}
$$

Since, by assumption, the quotient on the left hand side of Equation (6) is a polynomial, terms involving negative powers of x must cancel; therefore, these terms can be ignored.

To illustrate this division by truncated formal power series technique, consider the example:

$$\mu(x,y)/(x-y) = \frac{4x + 7x^2 - 4y - 7y^2}{x-y}.$$

Using the method, we replace the monomial x by the sum $\sum_{k=0}^{1-1} y^{1-1-k} x^k = 1$ and the monomial x^2 by the sum $\sum_{k=0}^{2-1} y^{2-1-k} x^k = y + x$. The quotient is then

$$\frac{4x + 7x^2 - 4y - 7y^2}{x-y} = 4 \times 1 + 7 \times (y + x)$$

as expected.

4 The Transformation Matrix From Sylvester to Bezout

If we perform the division in the Cayley expression (2) using the method of truncated formal power series and delay expanding $f(t)$ and $g(t)$ as sums, we obtain a relationship between the Sylvester matrix S_n and the Bezout matrix B_n. Recall that $L = [\, f(t) \;\; g(t) \,]$ and let $R_j = [\, b_j \;\; -a_j \,]^T$. Then

$$
\begin{aligned}
\Delta_n(t,\beta) &= (f(t) \sum_{j=0}^{n} b_j \beta^j - g(t) \sum_{j=0}^{n} a_j \beta^j)/(\beta - t) \\
&= \sum_{j=0}^{n} L R_j \beta^j / (\beta - t) \\
&= \sum_{j=0}^{n} L R_j \sum_{v=0}^{j-1} t^{j-1-v} \beta^v \\
&= \sum_{v=0}^{n-1} \sum_{j=v+1}^{n} L R_j t^{j-1-v} \beta^v. \qquad (7)
\end{aligned}
$$

Hence there is a matrix F_n such that

$$
\begin{aligned}
\Delta_n(t,\beta) &= \left[\; L \quad tL \quad \cdots \quad t^{n-1}L \;\right] F_n \begin{bmatrix} 1 \\ \vdots \\ \beta^{n-1} \end{bmatrix} \\
&= \left[\; 1 \quad \cdots \quad t^{2n-1} \;\right] S_n F_n \begin{bmatrix} 1 \\ \vdots \\ \beta^{n-1} \end{bmatrix} \qquad (8)
\end{aligned}
$$

Comparing Equation (8) with Equations (3) and (4), we see that

$$S_n F_n = \begin{bmatrix} B_n \\ 0_{n \times n} \end{bmatrix}.$$

There are n rows of zeros appended below B_n because, unlike S_n, B_n does not involve the monomials t^τ, $n \leq \tau \leq 2n-1$.

To find the entry of F_n indexed by $(t^\tau L, \beta^v)$, let $\tau = j-1-v$. Then $j = \tau+v+1$. Hence by Equation (7) the entry of F_n indexed by $(t^\tau L, \beta^v)$ is simply

$$R_{\tau+v+1}, \tag{9}$$

for $\tau+v+1 \leq n$ and zero otherwise. From this entry formula, we easily see that:

$$F_n = \begin{bmatrix} R_1 & R_2 & \cdots & R_{n-1} & R_n \\ R_2 & R_3 & \cdots & R_n & \\ \vdots & \vdots & \iddots & & \\ R_{n-1} & R_n & & & \\ R_n & & & & \end{bmatrix}, \tag{10}$$

since the entries are constant along the diagonals $\tau + v = constant$, and zero below the diagonal $\tau + v = n-1$. Though the dimension of F_n is $2n \times n$, F_n can be treated as a square matrix of dimension $n \times n$ if its entries are viewed as column vectors R_j. From this perspective, we see that F_n is striped, symmetric, and upper triangular.

5 Fast Computation of the Entries of the Bezout Resultant

An explicit entry formula for the Bezout matrix is provided in [Goldman *et al* 1984]. In this section, we derive this formula from the transformation matrix of Section 4. We also present a new, more efficient method for computing the entries of Bezout matrices and sums of Bezout matrices.

From Section 2 and Section 4, we know that

$$S_n = \begin{bmatrix} L_0 & & \\ \vdots & \ddots & \\ L_{n-1} & \cdots & L_0 \\ L_n & \cdots & L_1 \\ & \ddots & \vdots \\ & & L_n \end{bmatrix}, \qquad F_n = \begin{bmatrix} R_1 & \cdots & \cdots & R_n \\ \vdots & \iddots & \iddots & \\ \vdots & \iddots & & \\ R_n & & & \end{bmatrix},$$

and

$$S_n \cdot F_n = \begin{bmatrix} B_n \\ 0_{n \times n} \end{bmatrix}.$$

Hence

$$B_n = \begin{bmatrix} L_0 & & \\ \vdots & \ddots & \\ L_{n-1} & \cdots & L_0 \end{bmatrix} \cdot \begin{bmatrix} R_1 & \cdots & \cdots & R_n \\ \vdots & \iddots & \iddots & \\ \vdots & \iddots & & \\ R_n & & & \end{bmatrix}. \tag{11}$$

Write

$$B_n = \begin{bmatrix} Z_{0,0} & \cdots & Z_{0,n-1} \\ \vdots & & \vdots \\ Z_{n-1,0} & \cdots & Z_{n-1,n-1} \end{bmatrix}. \tag{12}$$

Then by Equation (11) and Equation (12),

$$Z_{i,j} = \sum_{l=0}^{\min(i,n-1-j)} L_{i-l} \cdot R_{j+1+l}, \qquad 0 \leq i, j \leq n-1. \tag{13}$$

Since $L_i \cdot R_j + L_j \cdot R_i = 0$, it is easy to verify that

$$\sum_{l=0}^{\max(0,i-j-1)} L_{i-l} \cdot R_{j+1+l} = 0.$$

Therefore, we can rewrite Equation (13) as

$$Z_{i,j} = \sum_{l=\max(0,i-j)}^{\min(i,n-1-j)} L_{i-l} \cdot R_{j+1+l}, \qquad 0 \leq i, j \leq n-1. \tag{14}$$

Equation (14) is equivalent to the explicit entry formula for the Bezout matrix given in [Goldman *et al* 1984]. By comparing the entry formulas in Equation (14), it follows that $Z_{i,j} = Z_{j,i}$. Thus we see again that B_n is a symmetric matrix.

From Equation (14), it is easy to recognize recursion along the diagonals $i+j=k$, for $0 \leq k \leq 2n-2$. In fact,

$$Z_{i,j} = Z_{i-1,j+1} + L_i \cdot R_{j+1}. \tag{15}$$

Equation (15) can be applied to compute the entries of B_n very efficiently using the following three-step algorithm:

1. Initialization:

$$(B_n)_{init} = \begin{bmatrix} L_0 \cdot R_1 & \cdots & L_0 \cdot R_n \\ & \ddots & \vdots \\ & & L_{n-1} \cdot R_n \end{bmatrix},$$

   that is,

$$(Z_{i,j})_{init} = L_i \cdot R_{j+1} = a_i b_{j+1} - a_{j+1} b_i, \qquad 0 \leq i \leq j \leq n-1.$$

2. Recursion:
   for $i = 1$ to $n-2$
   for $j = n-2$ to i (step -1)
   $Z_{i,j} \leftarrow Z_{i,j} + Z_{i-1,j+1}$.

   That is, we add along the diagonals from upper right to lower left, so schematically,

$$\begin{bmatrix} Z_{0,0} & \cdots & Z_{0,n-1} \\ & \ddots & \vdots \\ & & Z_{n-1,n-1} \end{bmatrix} = \begin{bmatrix} L_0 \cdot R_1 & L_0 \cdot R_2 & L_0 \cdot R_3 & \cdots & L_0 \cdot R_{n-3} & L_0 \cdot R_{n-2} & L_0 \cdot R_{n-1} \\ & \swarrow & \swarrow & \cdots & \swarrow & \swarrow & L_1 \cdot R_{n-1} \\ & & \swarrow & \cdots & \swarrow & \swarrow & L_2 \cdot R_{n-1} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & \swarrow & \swarrow & L_{n-3} \cdot R_{n-1} \\ & & & & & \swarrow & L_{n-2} \cdot R_{n-1} \\ & & & & & & L_{n-1} \cdot R_{n-1} \end{bmatrix}.$$

3. Symmetry: $Z_{i,j} = Z_{j,i}$, $i > j$.
   That is, the entries below the diagonal $i = j$ are obtained via symmetry from the entries above the diagonal.

Notice how much more efficient this recursive algorithm is compared to the explicit formula (Equation (14)). Using the explicit formula to find all the entries of B_n, we perform $O(n^3)$ additions and we recompute each product $L_i \cdot R_j$ many times. In the recursive algorithm we perform only $O(n^2)$ additions and we calculate each product $L_i \cdot R_j$ only once, thus removing redundant computations.

This method can also be applied to compute sums of Bezout matrices of the same size, say $n \times n$, efficiently. Instead of computing each Bezout matrix separately, and then adding them together, we can save time and space by adopting the following strategy of initialization and marching:

1. initialize the entries in this $n \times n$ matrix with indices $0 \le i \le j \le n-1$ using the sums of the initializations for each Bezout matrix—the previous initialize–and–march method assigns an initialization for each Bezout matrix in the sum;
2. march to the southwest along the diagonals $1 \le i+j \le 2n-3$.
3. obtain the entries with indices $i > j$ via symmetry.

In this way, we save not only the space for storing the individual Bezout matrices, but also we march along the diagonals just once; hence this approach greatly reduces the computational complexity. This fast computation of sums of Bezout matrices has applications in the efficient computation of resultants for bivariate polynomials [Chionh *et al* 1998].

6 Hybrid Resultants

A collection of $n-1$ hybrids of the Sylvester resultant and the Bezout resultant for f and g are constructed in [Weyman and Zelevinsky 1994; Sederberg *et al* 1997]. These hybrid resultants are composed of some columns from the Sylvester resultant and some columns from the Bezout resultant. In this section, we will generate these hybrids based on the transformation in Section 4 from the Sylvester to the Bezout resultant.

Let H_j, $j = 0, \cdots, n$, be the j–th hybrid resultant matrix

$$\begin{bmatrix} L_0 & & & Z_{0,0} & \cdots & Z_{0,j-1} \\ \vdots & \ddots & & \vdots & \vdots & \vdots \\ L_{n-j-1} & \ddots & L_0 & Z_{n-j-1,0} & \cdots & Z_{n-j-1,j-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ L_{n-1} & \ddots & L_j & Z_{n-1,0} & \cdots & Z_{n-1,j-1} \\ L_n & \ddots & L_{j+1} & & & \\ & \ddots & \vdots & & & \\ & & L_n & & & \end{bmatrix}.$$

That is, H_j contains the first $2(n-j)$ truncated columns of S_n, and the first j columns of B_n. Note that H_j is a square matrix of order $2n-j$; moreover, $H_0 = S_n$ and $H_n = B_n$. Below we will show that $|H_j| = \pm\, |H_{j+1}|$, $j = 0, \cdots, n-1$. Since S_n and B_n are known to be resultants for f and g, it follows that H_j is a resultant of f and g for $j = 0, \cdots, n$.

To proceed with our proof, let

$$R_j^i = \begin{bmatrix} R_i \\ \vdots \\ R_j \end{bmatrix},$$

$J = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and consider the $(2n-j) \times (2n-j)$ matrix

$$T_j = \begin{bmatrix} I_{2(n-j-1)} & 0 & R_{n-1}^{j+1} & 0 \\ 0 & 0 & R_n & J \\ 0 & I_j & 0 & 0 \end{bmatrix}.$$

Multiplying H_j by T_j, we get

$$
\begin{aligned}
H_j \cdot T_j &= \begin{bmatrix}
L_0 & & & Z_{0,0} & \cdots & Z_{0,j-1} \\
\vdots & \ddots & & \vdots & \vdots & \vdots \\
L_{n-j-1} & \ddots & L_0 & Z_{n-j-1,0} & \cdots & Z_{n-j+1,j-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
L_{n-1} & \ddots & L_j & Z_{n-1,0} & \cdots & Z_{n-1,j-1} \\
L_n & \ddots & L_{j+1} & & & \\
 & \ddots & \vdots & & & \\
 & & L_n & & &
\end{bmatrix} \cdot \begin{bmatrix}
I_{2(n-j-1)} & 0 & R_{n-1}^{j+1} & 0 \\
0 & 0 & R_n & J \\
0 & I_j & 0 & 0
\end{bmatrix} \\
&= \begin{bmatrix}
L_0 & & & Z_{0,0} & \cdots & Z_{0,j} & \\
\vdots & \ddots & & \vdots & \vdots & \vdots & \\
L_{n-j-2} & \ddots & L_0 & Z_{n-j-2,0} & \cdots & Z_{n-j-2,j} & \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & b_0 \\
L_{n-1} & \ddots & L_j & Z_{n-1,0} & \cdots & Z_{n-1,j} & \vdots \\
L_n & \ddots & L_{j+1} & & & & \vdots \\
 & \ddots & \vdots & & & & \vdots \\
 & & L_n & & & & b_{n-1} \\
 & & 0 & & & & b_n
\end{bmatrix}.
\end{aligned} \tag{16}
$$

Notice that the top-left $(2n-j-1) \times (2n-j-1)$ submatrix is exactly H_{j+1}, so Equation (16) can be rewritten as

$$H_j \cdot T_j = \begin{bmatrix} H_{j+1} & * \\ 0 & b_n \end{bmatrix}. \tag{17}$$

Therefore,

$$|H_j| \cdot |T_j| = |H_{j+1}| \cdot b_n$$

But by construction,

$$
\begin{aligned}
|T_j| &= \pm \, |\,[\, R_n \quad J\,]\,| \\
&= \pm \begin{vmatrix} b_n & 0 \\ -a_n & 1 \end{vmatrix} \\
&= \pm \, b_n,
\end{aligned}
$$

so

$$|H_j| = \pm \, |H_{j+1}|. \tag{18}$$

Thus we generate a sequence of resultants that are hybrids of the Sylvester and Bezout resultants. Since $H_0 = S_n$ and $H_n = B_n$, we obtain in this fashion a direct proof that

$$|S_n| = \pm \, |B_n| \tag{19}$$

without appealing to any specific properties of resultants.

7 Non-Homogeneous Bezout Matrices Due to Unequal Degrees

When the coefficients a_i, b_i, $0 \le i \le n$, of f and g are treated as formal symbols, the Bezout matrix B_n is homogeneous in the sense that each entry is quadratic in the a_i's and b_i's.

If the degree of g is m and $m < n$, then we have $b_{m+1} = \cdots = b_n = 0$. In this situation, $|S_n|$ is not exactly the resultant of f, g—it has an extraneous factor of $(a_n)^{n-m}$. Rather the correct Sylvester matrix $S_{n,m}$, whose determinant is the exact resultant, is the coefficient matrix of $n+m$ polynomials:

$$\begin{bmatrix} L & tL & \cdots & t^{m-1}L & t^m g & \cdots & t^{n-1}g \end{bmatrix} = \begin{bmatrix} 1 & \cdots & t^{n+m-1} \end{bmatrix} S_{n,m}. \tag{20}$$

Note that S_n is of order $2n$, while $S_{n,m}$ is of order $n+m$, and

$$|S_n| = \pm\,(a_n)^{n-m} \cdot |S_{n,m}|. \tag{21}$$

Similarly, the Bezout matrix B_n obtained from the Cayley expression $\Delta_n(t,\beta)$ also contains the extraneous factor a_n^{n-m}. It is pointed out in [Cox *et al* 1997] that in this case the correct Bezout resultant for f and g can be written as the determinant of a non-homogeneous matrix $B_{n,m}$ of size $n \times n$. The matrix $B_{n,m}$ has m columns of quadratic entries consisting of coefficients of both f and g, and $n-m$ columns of linear entries consisting of coefficients of g. Below we present a simple alternative proof of this fact.

Recall from Section 4 that

$$\begin{aligned}
\begin{bmatrix} B_n \\ 0 \end{bmatrix} &= S_n \cdot F_n = S_n \cdot \begin{bmatrix} R_1 & \cdots & R_n \\ \vdots & \iddots & \\ R_n & & \end{bmatrix} \\
&= \left[S_n \cdot \begin{bmatrix} R_1 & \cdots & R_m \\ \vdots & \iddots & \vdots \\ R_{n-m+1} & \iddots & R_n \\ \vdots & \iddots & \\ R_n & & \end{bmatrix} \quad S_n \cdot \begin{bmatrix} R_{m+1} & \cdots & R_n \\ \vdots & \iddots & \vdots \\ R_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \right].
\end{aligned} \tag{22}$$

Let us focus on the second matrix product on the right hand side of Equation (22). With $b_{m+1} = \cdots = b_n = 0$,

$$\begin{aligned}
S_n \cdot \begin{bmatrix} R_{m+1} & \cdots & R_n \\ \vdots & \iddots & \vdots \\ R_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} &= \begin{bmatrix} L_0 & & \\ \vdots & \ddots & \\ L_{n-m-1} & \cdots & L_0 \\ \vdots & \vdots & \vdots \\ L_{n-1} & \cdots & L_m \\ L_n & \cdots & L_{m+1} \\ & \ddots & \vdots \\ & & L_n \\ & & 0_{m\times 2} \end{bmatrix} \cdot \begin{bmatrix} 0 & & 0 \\ -a_{m+1} & \cdots & -a_n \\ \vdots & \iddots & \\ 0 & & \\ -a_n & & \end{bmatrix} \\
&= \begin{bmatrix} b_0 & & \\ \vdots & \ddots & \\ b_{n-m-1} & \cdots & b_0 \\ \vdots & \vdots & \vdots \\ b_{n-1} & \cdots & b_m \\ b_n & \cdots & b_{m+1} \\ & \ddots & \vdots \\ & & b_n \\ & & 0_{m\times 1} \end{bmatrix} \cdot \begin{bmatrix} -a_{m+1} & \cdots & -a_n \\ \vdots & \iddots & \\ -a_n & & \end{bmatrix}.
\end{aligned} \tag{23}$$

Note that the term $0_{m\times 2}$ in the first factor on the right hand side of Equation (23) represents an $m \times 2$

matrix of zero entries, since S_n has $2n$ rows. Let

$$A_{m,n} = \begin{bmatrix} -a_{m+1} & \cdots & -a_n \\ \vdots & \iddots & \\ -a_n & & \end{bmatrix}.$$

Then putting together Equations (22) and (23), and using the notation of Equation (12), we obtain

$$B_n = \left[\begin{bmatrix} Z_{0,0} & \cdots & Z_{0,m-1} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ Z_{n-1,0} & \cdots & Z_{n-1,m-1} \end{bmatrix} \begin{bmatrix} b_0 & & \\ \vdots & \ddots & \\ b_m & \ddots & \ddots & \\ & \ddots & \ddots & b_0 \\ & & \ddots & \vdots \\ & & & b_m \end{bmatrix} \right] \cdot \begin{bmatrix} I_m & \\ & A_{m,n} \end{bmatrix}. \tag{24}$$

Let us denote the $n \times n$ matrix in the first factor on the right hand side of Equation (24) by $B_{n,m}$. Then we can rewrite Equation (24) as

$$B_n = B_{n,m} \cdot \begin{bmatrix} I_m & \\ & A_{m,n} \end{bmatrix}. \tag{25}$$

Hence

$$|B_n| = \pm\, (-a_n)^{n-m} \cdot |B_{n,m}|. \tag{26}$$

Now it follows from Equations (19), (21), (26) that

$$|S_{n,m}| = \pm\, |B_{n,m}|. \tag{27}$$

That is, the coefficient matrix of the following n polynomials is the Bezout matrix for f and g:

$$\begin{bmatrix} D_0 & D_1 & \cdots & D_{m-1} & g & \cdots & t^{n-m-1}g \end{bmatrix}. \tag{28}$$

For example, we have

$$B_{5,2} = \begin{bmatrix} |0,1| & |0,2| & b_0 & & \\ |0,2| & |0,3|+|1,2| & b_1 & b_0 & \\ |0,3| & |0,4|+|1,3| & b_2 & b_1 & b_0 \\ |0,4| & |0,5|+|1,4| & & b_2 & b_1 \\ |0,5| & |1,5| & & & b_2 \end{bmatrix},$$

$$B_{5,3} = \begin{bmatrix} |0,1| & |0,2| & |0,3| & b_0 & \\ |0,2| & |0,3|+|1,2| & |0,4|+|1,3| & b_1 & b_0 \\ |0,3| & |0,4|+|1,3| & |0,5|+|1,4|+|2,3| & b_2 & b_1 \\ |0,4| & |0,5|+|1,4| & |1,5|+|2,4| & b_3 & b_2 \\ |0,5| & |1,5| & |2,5| & & b_3 \end{bmatrix},$$

where $|i,j| = L_i \cdot R_j = a_i b_j - a_j b_i$, $0 \le i,j \le n$.

In summary, there exist matrices $F_n(m)$ and $F_{n,m}$ such that

$$S_n \cdot F_n(m) = \begin{bmatrix} B_{n,m} \\ 0 \end{bmatrix} = S_{n,m} \cdot F_{n,m}.$$

In fact, let $J = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$; then

$$F_n(m) = \begin{bmatrix} R_1 & \cdots & R_m & J & & & \\ R_2 & \cdots & R_{m+1} & & J & & \\ \vdots & \iddots & \vdots & & & \ddots & \\ R_{n-m} & \cdots & R_{n-1} & & & & J \\ R_{n-m+1} & \iddots & R_n & & & & \\ \vdots & \iddots & & & & & \\ R_n & & & & & & \end{bmatrix}.$$

Moreover, when $n - m > m$,

$$F_{n,m} = \begin{bmatrix} R_1 & \cdots & R_m & J & & & & & \\ \vdots & \iddots & \vdots & & \ddots & & & & \\ R_m & \cdots & R_{2m-1} & & & J & & & \\ -a_{m+1} & \cdots & -a_{2m} & & & & 1 & & \\ \vdots & \iddots & \vdots & & & & & \ddots & \\ -a_{n-m} & \cdots & -a_{n-1} & & & & & & 1 \\ -a_{n-m+1} & \cdots & -a_n & & & & & & \\ \vdots & \iddots & & & & & & & \\ -a_n & & & & & & & & \end{bmatrix},$$

and when $n - m \leq m$,

$$F_{n,m} = \begin{bmatrix} R_1 & \cdots & R_{n-m} & \cdots & R_m & J & & \\ \vdots & \iddots & \vdots & \iddots & \vdots & & \ddots & \\ R_{n-m} & \cdots & R_{2n-2m-1} & \cdots & R_{n-1} & & & J \\ R_{n-m+1} & \cdots & R_{2n-2m} & \cdots & R_n & & & \\ \vdots & \iddots & \vdots & \iddots & & & & \\ R_m & \cdots & R_{n-1} & & & & & \\ -a_{m+1} & \cdots & -a_n & & & & & \\ \vdots & \iddots & & & & & & \\ -a_n & & & & & & & \end{bmatrix}.$$

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