The Block Structure of Three Dixon Resultants and Their Accompanying Transformation Matrices

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Abstract

Dixon [1908] introduces three distinct determinant formulations for the resultant of three bivariate polynomials of bidegree (m, n). The first technique applies Sylvester's dialytic method to construct the resultant as the determinant of a matrix of order $6mn$. The second approach uses Cayley's determinant device to form a more compact representation for the resultant as the determinant of a matrix of order $2mn$. The third method employs a combination of Cayley's determinant device with Sylvester's dialytic method to build the resultant as the determinant of a matrix of order $3mn$. Here relations between these three resultant formulations are derived and the structure of the transformations between these resultant matrices is investigated. In particular, it is shown that these transformation matrices all have similar, simple, upper triangular, block symmetric structures and the blocks themselves have elegant symmetry properties. Elementary entry formulas for the transformation matrices are also provided. In light of these results, the three Dixon resultant matrices are reexamined and shown to have natural block structures compatible with the block structures of the transformation matrices. These block structures are analyzed here and applied along with the block structures of the transformation matrices to simplify the calculation of the entries of the Dixon resultants of order $2mn$ and $3mn$ and to make these calculations more efficient by removing redundant computations.

1 Introduction

Dixon [1908] describes three distinct determinant expressions for the resultant of three bivariate polynomials of bidegree (m, n). The first formulation is the determinant of a $6mn \times 6mn$ matrix generated by Sylvester's dialytic method. We shall refer to this determinant as the *Sylvester resultant*. The second expression is the determinant of a $2mn \times 2mn$ matrix found by using an extension of Cayley's determinant device for generating the Bezout resultant for two univariate polynomials. This $2mn \times 2mn$ matrix has come to be known as the Dixon resultant [Sederberg, 1983; Sederberg *et al* 1984; Chionh, 1990; Manocha and Canny, 1992]. However, to distinguish this determinant expression from Dixon's other two formulations for the resultant, we shall call this determinant the *Cayley resultant*. The third representation is the determinant of a $3mn \times 3mn$ matrix generated by combining Cayley's determinant device with Sylvester's dialytic method. We shall refer to this determinant as the *mixed Cayley-Sylvester resultant*.

Dixon introduces these three representations for the resultant independently, without attempting to relate one to the other. The purpose of this paper is to derive connections between these three distinct representations for the resultant. In particular, we will prove that the polynomials represented by the columns of the Cayley and the mixed Cayley-Sylvester resultants are linear combinations of the polynomials represented by the columns of the Sylvester resultant. Thus there are matrices relating the Sylvester resultant to the Cayley and mixed Cayley-Sylvester resultants. We shall show that these transformation matrices all have similar, simple, upper triangular, block symmetric structures and the blocks themselves are either Sylvester-like or symmetic. In addition, we shall provide straightforward formulas for the entries of these matrices.

Reexamining the three Dixon resultant matrices, we find that they too have simple block structures compatible with the block structures of the transformation matrices. Indeed, we shall see that the blocks for the transformation matrices are essentially transposes of the blocks for these resultant matrices, although the arrangements of the blocks within these matrices differ. We shall also show that the blocks of the Sylvester and mixed Cayley-Sylvester matrices are related to the Sylvester and Bezout resultants of certain univariate polynomials.

The entries of the Sylvester resultant are just the coefficients of the original polynomials, but the entries of the Cayley and mixed Cayley-Sylvester resultants are more complicated expressions in these coefficients. We shall show how to take advantage of the block structure of the resultant matrices together with the block structure of the transformation matrices to simplify the calculation of the entries of the Cayley and mixed Cayley-Sylvester resultants and to remove redundant computations. These techniques for the efficient computation of the entries of the Cayley and mixed Cayley-Sylvester resultants are extensions to the bivariate setting of similar results on the efficient computation of the entries of the Bezout resultant of two univariate polynomials [Chionh *et al* 1998].

In a follow up paper [Zhang *et al* 1998] we plan to derive a new class of representations for the bivariate resultant in terms of determinants of hybrid matrices formed by mixing and matching appropriate columns from the Sylvester, Cayley, or mixed Cayley-Sylvester matrices. The derivation of these hybrid resultants is also based on the block structure of the Dixon resultants and the accompanying transformation matrices derived in this paper. Again these new hybrid resultants are generalizations to the bivariate setting of known hybrids between the Sylvester and Bezout resultants for two univariate polynomials [Weyman and Zelevinsky 1994; Sederberg *et al* 1997; Chionh *et al* 1998].

This paper is structured in the following manner. We begin in Section 2 by establishing our notation and reviewing Dixon's formulations of the Sylvester, Cayley, and mixed Cayley-Sylvester resultants. Next, in Section 3, we describe the truncated formal power series technique that is essential for the derivation of the matrices relating these three Dixon resultants. We go on in Sections 4 and 5 to derive explicit formulas for the entries of the matrices relating the Sylvester resultant to the Cayley and mixed Cayley-Sylvester resultants and to discuss the block structure and symmetry properties of these transformation matrices. We devote Section 6 to deriving similar results for the transformation relating the mixed Cayley-Sylvester and Cayley resultants. We close in Section 7 by analyzing the block structure of the three Dixon resultants and showing how to take advantage of this block structure together with the block structure of the transformation matrices to simplify the calculation of the entries of the Cayley and mixed Cayley-Sylvester resultants and to make the calculations more efficient by removing redundant computations. We also derive some remarkable convolution identities relating the blocks of the Sylvester and the blocks of the mixed Cayley-Sylvester resultants.

2 The Three Dixon Resultants

Consider three bivariate polynomials of bidegree (m, n):

$$f(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j} s^i t^j, \qquad g(s,t) = \sum_{k=0}^{m}\sum_{l=0}^{n} b_{k,l} s^k t^l, \qquad h(s,t) = \sum_{p=0}^{m}\sum_{q=0}^{n} c_{p,q} s^p t^q.$$

Dixon outlines three methods for constructing a resultant for f, g, h [Dixon 1908]. In this section, we briefly review the construction of each of Dixon's resultants.

2.1 The Sylvester Resultant

The Sylvester resultant for $f(s,t), g(s,t), h(s,t)$ is constructed using Sylvester's dialytic method. Consider the $6mn$ polynomials $\{s^\sigma t^\tau f, s^\sigma t^\tau g, s^\sigma t^\tau h \mid \sigma = 0, \cdots, 2m-1; \tau = 0, \cdots, n-1\}$. Let $L(s,t) = \begin{bmatrix} f(s,t) & g(s,t) & h(s,t) \end{bmatrix}$; then this system of polynomials can be written in matrix notation as

$$\begin{bmatrix} L & \cdots & s^\sigma t^\tau L & s^\sigma t^{\tau+1} L & \cdots & s^{2m-1} t^{n-1} L \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{3m-1} t^{2n-1} \end{bmatrix}^T S_{m,n}, \tag{1}$$

where the rows are indexed lexicographically with $s > t$. That is, the monomials are ordered as $1, t, \cdots, t^{2n-1}, \cdots, s^\sigma, s^\sigma t, \cdots, s^\sigma t^{2n-1}, \cdots, s^{3m-1}, s^{3m-1} t, \cdots, s^{3m-1} t^{2n-1}$. Notice that the coefficient matrix $S_{m,n}$ is a square matrix of order $6mn$. The Sylvester resultant for f, g, h is simply $|S_{m,n}|$.

2.2 The Cayley Resultant

The Cayley resultant for f, g, h can be derived from the Cayley expression

$$\Delta_{m,n}(s,t,\alpha,\beta) = \frac{\begin{vmatrix} f(s,t) & g(s,t) & h(s,t) \\ f(\alpha,t) & g(\alpha,t) & h(\alpha,t) \\ f(\alpha,\beta) & g(\alpha,\beta) & h(\alpha,\beta) \end{vmatrix}}{(\alpha - s)(\beta - t)}. \tag{2}$$

Since the numerator vanishes when $\alpha = s$ or $\beta = t$, the numerator is divisible by $(\alpha - s)(\beta - t)$. Hence $\Delta_{m,n}(s,t,\alpha,\beta)$ is a polynomial in s, t, α, β, so

$$\Delta_{m,n}(s,t,\alpha,\beta) = \sum_{u=0}^{2m-1} \sum_{v=0}^{n-1} \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{2n-1} c_{\sigma,\tau,u,v} s^\sigma t^\tau \alpha^u \beta^v.$$

In matrix form,

$$\Delta_{m,n}(s,t,\alpha,\beta) = \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1} t^{2n-1} \end{bmatrix}^T C_{m,n} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{2m-1} \beta^{n-1} \end{bmatrix}. \tag{3}$$

The rows and columns of $C_{m,n}$ are indexed lexicographically by the pairs $(s^\sigma t^\tau, \alpha^u \beta^v)$ with $s > t$, $\alpha > \beta$. Notice that the coefficient matrix $C_{m,n}$ is again a square matrix but of order $2mn$. The Cayley resultant for f, g, h is $|C_{m,n}|$.

2.3 The Mixed Cayley-Sylvester Resultant

The mixed Cayley-Sylvester resultant can be derived by combining Cayley's determinant device with Sylvester's dialytic method. Consider the expression

$$\phi(g,h) = \begin{vmatrix} g(s,t) & h(s,t) \\ g(s,\beta) & h(s,\beta) \end{vmatrix} / (\beta - t). \tag{4}$$

Since the numerator is always divisible by the denominator, $\phi(g,h)$ is a polynomial in s, t, β, with degree $2m$ in s, $n-1$ in t, and $n-1$ in β. Collecting the coefficients of β^j, $j = 0, \cdots, n-1$, we get n polynomials $\overline{f}_j(s,t)$ such that $\phi(g,h) = \sum_{j=0}^{n-1} \overline{f}_j(s,t)\beta^j$. Multiplying these n polynomials by the m monomials $1, s, \cdots, s^{m-1}$, we obtain mn polynomials $s^i \overline{f}_j(s,t)$, $0 \le i \le m-1$, $0 \le j \le n-1$, in the $3mn$ monomials $s^i t^j$, $0 \le i \le 3m-1$, $0 \le j \le n-1$.

Now do the same for

$$\phi(h,f) = \begin{vmatrix} h(s,t) & f(s,t) \\ h(s,\beta) & f(s,\beta) \end{vmatrix} /(\beta - t), \qquad \phi(f,g) = \begin{vmatrix} f(s,t) & g(s,t) \\ f(s,\beta) & g(s,\beta) \end{vmatrix} /(\beta - t).$$

Altogether we get $3mn$ polynomials $s^i \overline{f}_j, s^i \overline{g}_j, s^i \overline{h}_j$, $j = 0, \cdots, n-1$, $i = 0, \cdots, m-1$, in $3mn$ monomials, where

$$\phi(g,h) = \sum_{j=0}^{n-1} \overline{f}_j(s,t)\beta^j, \qquad \phi(h,f) = \sum_{j=0}^{n-1} \overline{g}_j(s,t)\beta^j, \qquad \phi(f,g) = \sum_{j=0}^{n-1} \overline{h}_j(s,t)\beta^j.$$

Let $\psi(s,\alpha) = \sum_{u=0}^{m-1} s^u \alpha^u$ and $\overline{L}_k = [\, \overline{f}_k \quad \overline{g}_k \quad \overline{h}_k \,]$. Then

$$\begin{aligned}
&\psi(s,\alpha) \left[\ \phi(g,h) \quad \phi(h,f) \quad \phi(f,g)\ \right] \\
&= [\, \overline{L}_0 \quad \cdots \quad \overline{L}_{n-1} \quad \cdots \quad s^{m-1}\overline{L}_0 \quad \cdots \quad s^{m-1}\overline{L}_{n-1} \,] \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{m-1}\beta^{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{3m-1} t^{n-1} \end{bmatrix}^T M_{m,n} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{m-1}\beta^{n-1} \end{bmatrix}, \qquad (5)
\end{aligned}$$

where $M_{m,n}$ is the coefficient matrix of these $3mn$ polynomials. Note that in this notation, $M_{m,n}$ has $3mn$ rows and mn columns, but each entry of $M_{m,n}$ is actually a 1×3 matrix. Thus the real size of $M_{m,n}$ is $3mn \times 3mn$; hence $M_{m,n}$ is a square matrix. The mixed Cayley-Sylvester resultant is $|M_{m,n}|$.

2.4 Notation

We plan to derive explicit formulas for the entries of the conversion matrices between $S_{m,n}$, $C_{m,n}$ and $M_{m,n}$, and then to apply these formulas to elucidate the structure and simplify the computation of the entries of the three Dixon resultants. For convenience, we adopt the following notation:

$$|i,j;k,l;p,q| = \begin{vmatrix} a_{i,j} & b_{i,j} & c_{i,j} \\ a_{k,l} & b_{k,l} & c_{k,l} \\ a_{p,q} & b_{p,q} & c_{p,q} \end{vmatrix},$$

$$A_{k,l;p,q} = \begin{vmatrix} b_{k,l} & c_{k,l} \\ b_{p,q} & c_{p,q} \end{vmatrix}, \qquad B_{k,l;p,q} = \begin{vmatrix} c_{k,l} & a_{k,l} \\ c_{p,q} & a_{p,q} \end{vmatrix}, \qquad C_{k,l;p,q} = \begin{vmatrix} a_{k,l} & b_{k,l} \\ a_{p,q} & b_{p,q} \end{vmatrix},$$

$$L_{i,j} = [\, a_{i,j} \quad b_{i,j} \quad c_{i,j} \,], \qquad R_{k,l,p,q} = [\, A_{k,l;p,q} \quad B_{k,l;p,q} \quad C_{k,l;p,q} \,],$$

$$X_{p,q} = \begin{bmatrix} 0 & -c_{p,q} & b_{p,q} \\ c_{p,q} & 0 & -a_{p,q} \\ -b_{p,q} & a_{p,q} & 0 \end{bmatrix}.$$

2.5 Example: $m = n = 1$

$$\begin{aligned}
S_{1,1} &= \begin{bmatrix} a_{0,0} & b_{0,0} & c_{0,0} & 0 & 0 & 0 \\ a_{0,1} & b_{0,1} & c_{0,1} & 0 & 0 & 0 \\ a_{1,0} & b_{1,0} & c_{1,0} & a_{0,0} & b_{0,0} & c_{0,0} \\ a_{1,1} & b_{1,1} & c_{1,1} & a_{0,1} & b_{0,1} & c_{0,1} \\ 0 & 0 & 0 & a_{1,0} & b_{1,0} & c_{1,0} \\ 0 & 0 & 0 & a_{1,1} & b_{1,1} & c_{1,1} \end{bmatrix} = \begin{bmatrix} L_{0,0} & 0 \\ L_{0,1} & 0 \\ L_{1,0} & L_{0,0} \\ L_{1,1} & L_{0,1} \\ 0 & L_{1,0} \\ 0 & L_{1,1} \end{bmatrix}, \\
C_{1,1} &= \begin{bmatrix} |0,0;1,0;0,1| & -|1,0;0,0;1,1| \\ |0,0;1,1;0,1| & -|1,0;0,1;1,1| \end{bmatrix}, \\
M_{1,1} &= \begin{bmatrix} A_{0,0;0,1} & B_{0,0;0,1} & C_{0,0;0,1} \\ A_{0,0;1,1} + A_{1,0;0,1} & B_{0,0;1,1} + B_{1,0;0,1} & C_{0,0;1,1} + C_{1,0;0,1} \\ A_{1,0;1,1} & B_{1,0;1,1} & C_{1,0;1,1} \end{bmatrix} = \begin{bmatrix} R_{0,0;0,1} \\ R_{0,0;1,1} + R_{1,0;0,1} \\ R_{1,0;1,1} \end{bmatrix}.
\end{aligned}$$

2.6 Review and Preview

In this section, we have introduced the three Dixon determinants for the resultant of three bivariate polynomials of bidegree (m, n). In Sections 4, 5, 6, we will investigate relationships between the three Dixon resultant matrices $S_{m,n}$, $C_{m,n}$, and $M_{m,n}$. It follows by inspection from Equations (1), (3), (5) that the columns of Dixon's three resultant matrices represent bivariate polynomials in s, t. In particular, the columns of $S_{m,n}$, $M_{m,n}$, $C_{m,n}$ are respectively bivariate polynomials of bidegree $(3m-1, 2n-1)$, $(3m-1, n-1)$, $(m-1, 2n-1)$. Since the columns of $S_{m,n}$ are linearly independent, the $6mn$ polynomials represented by these columns span the space of bivariate polynomials of bidegree $(3m-1, 2n-1)$. Therefore the polynomials represented by the columns of $M_{m,n}$ and $C_{m,n}$ must be linear combinations of the polynomials represented by the columns of $S_{m,n}$. Sections 4, 5 are devoted to deriving explicit formulas for the transformations from $S_{m,n}$ to $M_{m,n}$ and $C_{m,n}$ compatible with the polynomial indexing of the rows and columns.

3 The Technique of Truncated Formal Power Series

The key technique that we shall harness to derive explicit relationships between the three Dixon resultant matrices is the method of truncated formal power series. To illustrate the method, let us consider a simple example. If we write

$$\frac{1}{A-B} = \frac{1}{A}\left(1 - \frac{B}{A}\right)^{-1} = \sum_{i=0}^{\infty} \frac{B^i}{A^{i+1}}$$

we can turn the division

$$\frac{A^2 - B^2}{A - B}$$

into the multiplication

$$A^2\left(\frac{1}{A} + \frac{B}{A^2}\right) = A + B.$$

This transformation from division to multiplication is permissible when we know a priori that the denominator exactly divides the numerator. In this case, the quotient has no negative powers, so the result of the division is simply a truncated finite sum of products. The product of a numerator term and a power series term need be considered only if the product involves no negative powers. Only such products can survive when the rational expression is actually a polynomial.

In Dixon's constructions, there are various rational expressions involving division by $\alpha - s$ or $\beta - t$. Since these rational expressions are actually polynomials, we can apply the technique of truncated formal power series to change division into multiplication. If $\nu(x, y)$ is a polynomial in x and y,

$$\nu(x,y) = \sum_{i=0}^{m} a_i(y)x^i = \sum_{j=0}^{n} b_j(x)y^j,$$

which is divisible by $x - y$, then we have

$$\frac{\nu(x,y)}{x-y} = \sum_{i=0}^{m}\sum_{k=0}^{i-1} a_i(y)y^{i-1-k}x^k = -\sum_{j=0}^{n}\sum_{k=0}^{j-1} b_j(x)x^{j-1-k}y^k.$$

Note that when i or j is zero the summation range of k is vacuous. For brevity we usually treat a sum of vacuous range as zero and we will not adjust the summation range to be non-vacuous.

4 From Sylvester to Mixed Cayley-Sylvester

Recall that the columns of $S_{m,n}$ and $M_{m,n}$ represent respectively bivariate polynomials of bidegree $(3m-1, 2n-1)$ and $(3m-1, n-1)$. In order to derive a $6mn \times 3mn$ conversion matrix $G_{m,n}$ such that

$$S_{m,n}G_{m,n} = \begin{bmatrix} M_{m,n} \\ 0 \end{bmatrix}, \tag{6}$$

that is, in order to insure that the $3mn$ zero rows are not interleaved with the $3mn$ non-zero rows, we have to abandon the default lexicographic orders $s > t$, $\alpha > \beta$ and impose instead the lexicographic orders $t > s$, $\beta > \alpha$. Matrices with this order of rows and columns will be indicated by an "$*$" to differentiate them from those matrices with the default lexicographic orders. The derivation below of the transformation matrix produces an explicit entry formula that reveals the simple elegant block structure of $G^*_{m,n}$.

4.1 The Conversion Matrix $G^*_{m,n}$

To find $G^*_{m,n}$, we expand the numerator on the right hand side of Equation (4) by cross-multiplying and writing $g(s,\beta)$, $h(s,\beta)$ as explicit sums of monomials to obtain:

$$\phi(g,h) = \sum_{p=0}^{m}\sum_{q=0}^{n}(g(s,t)c_{p,q} - h(s,t)b_{p,q})s^p\beta^q/(\beta - t).$$

Applying the technique of truncated formal power series with the numerator as a polynomial in β yields:

$$\phi(g,h) = \sum_{p=0}^{m}\sum_{q=0}^{n}(g(s,t)c_{p,q} - h(s,t)b_{p,q})s^p\sum_{v=0}^{q-1} t^{q-1-v}\beta^v.$$

Rearranging the range of q and v results in:

$$\phi(g,h) = \sum_{v=0}^{n-1}\sum_{p=0}^{m}\sum_{q=v+1}^{n}(g(s,t)c_{p,q} - h(s,t)b_{p,q})s^p t^{q-1-v}\beta^v.$$

Substituting $L = \left[\begin{array}{ccc} f & g & h \end{array}\right]$ leads to:

$$\phi(g,h) = \sum_{v=0}^{n-1}\sum_{p=0}^{m}\sum_{q=v+1}^{n} L \left[\begin{array}{c} 0 \\ c_{p,q} \\ -b_{p,q} \end{array}\right] s^p t^{q-1-v} \beta^v.$$

Hence

$$\begin{aligned} \psi(s,\alpha)\phi(g,h) &= \sum_{u=0}^{m-1} s^u \alpha^u \phi(g,h), \\ &= \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\sum_{p=0}^{m}\sum_{q=v+1}^{n} L \left[\begin{array}{c} 0 \\ c_{p,q} \\ -b_{p,q} \end{array}\right] s^{p+u} t^{q-1-v} \alpha^u \beta^v. \end{aligned}$$

Recalling that

$$X_{p,q} = \left[\begin{array}{ccc} 0 & -c_{p,q} & b_{p,q} \\ c_{p,q} & 0 & -a_{p,q} \\ -b_{p,q} & a_{p,q} & 0 \end{array}\right],$$

we have

$$\psi(s,\alpha)\left[\begin{array}{ccc} \phi(g,h) & \phi(h,f) & \phi(f,g) \end{array}\right] = \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\sum_{p=0}^{m}\sum_{q=v+1}^{n} L X_{p,q} s^{p+u} t^{q-1-v} \alpha^u \beta^v. \tag{7}$$

Since $0 \le p+u \le 2m-1$ and $0 \le q-1-v \le n-1$, this equation can be written as

$$\psi(s,\alpha)\left[\begin{array}{ccc} \phi(g,h) & \phi(h,f) & \phi(f,g) \end{array}\right] = \left[\begin{array}{c} L^T \\ \vdots \\ t^\tau s^\sigma L^T \\ t^\tau s^{\sigma+1} L^T \\ \vdots \\ t^{n-1} s^{2m-1} L^T \end{array}\right]^T G^*_{m,n} \left[\begin{array}{c} 1 \\ \vdots \\ \beta^v \alpha^u \\ \beta^v \alpha^{u+1} \\ \vdots \\ \beta^{n-1}\alpha^{m-1} \end{array}\right], \tag{8}$$

where $G^*_{m,n}$ is the $2mn \times mn$ matrix whose entries are the 3×3 matrices $X_{p,q}$. Thus the actual dimensions of $G^*_{m,n}$ are $6mn \times 3mn$.

It follows by the definition of $S^*_{m,n}$ (c.f. Equation (1) with the new lexicographic order) that

$$\psi(s,\alpha)\left[\begin{array}{ccc} \phi(g,h) & \phi(h,f) & \phi(f,g) \end{array}\right] = \left[\begin{array}{c} 1 \\ \vdots \\ t^\tau s^\sigma \\ t^\tau s^{\sigma+1} \\ \vdots \\ t^{2n-1} s^{3m-1} \end{array}\right]^T S^*_{m,n} G^*_{m,n} \left[\begin{array}{c} 1 \\ \vdots \\ \beta^v \alpha^u \\ \beta^v \alpha^{u+1} \\ \vdots \\ \beta^{n-1}\alpha^{m-1} \end{array}\right]. \tag{9}$$

Therefore, by the definition of $M^*_{m,n}$ (c.f. Equation (5) with the new lexicographic order), we see immediately that

$$S^*_{m,n} \cdot G^*_{m,n} = \left[\begin{array}{c} M^*_{m,n} \\ 0 \end{array}\right]. \tag{10}$$

4.2 The Entries of $G^*_{m,n}$

By solving $p+u=\sigma$, $q-1-v=\tau$ in Equation (7), we obtain $p=\sigma-u$, $q=\tau+v+1$. Consequently the 3×3 entry $X_{p,q}$ of $G^*_{m,n}$ indexed by $(t^\tau s^\sigma, \beta^v \alpha^u)$ is

$$X_{\sigma-u,\tau+v+1} \tag{11}$$

for $0 \le \sigma - u \le m$ and $\tau + v + 1 \le n$; the entries are zero everywhere else.

4.3 Properties of $G^*_{m,n}$

The structure of the $6mn \times 3mn$ conversion matrix $G^*_{m,n}$ is most revealing if we view it as a $2mn \times mn$ matrix whose entries are the 3×3 matrices $X_{p,q}$. Let $B^*_{\tau,v}$ be the $2m \times m$ submatrix of $G^*_{m,n}$ indexed by $(t^\tau s^\sigma, \beta^v \alpha^u)$, $0 \le \sigma \le 2m-1$, $0 \le u \le m-1$. That is, $B^*_{\tau,v}$ contains the matrices $X_{\sigma-u,\tau+v+1}$ such that $0 \le \sigma \le 2m-1$, $0 \le u \le m-1$. Using the entry formula (11), we readily observe the following properties.

4.3.1 $G^*_{m,n}$ is Block Upper Triangular

The blocks $B^*_{\tau,v}$ along the lower-left/top-right diagonal have $\tau + v = n-1$. The blocks $B^*_{\tau,v}$ below this diagonal have $\tau + v + 1 > n$ and hence are zero by the entry formula (11).

4.3.2 $G^*_{m,n}$ is Block Symmetric and Striped

By the entry formula (11) it follows easily that $B^*_{\tau,v} = B^*_{v,\tau}$ and $B^*_{\tau+1,v-1} = B^*_{\tau,v} = B^*_{\tau-1,v+1}$. Hence $G^*_{m,n}$ is block symmetric and $G^*_{\tau+v} = B^*_{\tau,v}$ is well-defined. Consequently,

$$G^*_{m,n} = \begin{bmatrix} G^*_0 & G^*_1 & \cdots & G^*_{n-1} \\ G^*_1 & \iddots & \iddots & \\ \vdots & \iddots & & \\ G^*_{n-1} & & & \end{bmatrix}. \tag{12}$$

4.3.3 The Lower-Left and Upper-Right Corners of G^*_i are Zeros

The $(m-1)m/2$ entries at the lower-left and the $(m-1)m/2$ entries at the upper-right corners of G^*_i are indexed by (σ, u) with $\sigma > u + m$ and $u > \sigma$ respectively; hence they are zero by the entry formula (11).

4.3.4 The Columns of G^*_i March Southeasterly

By the entry formula (11) we have $G^*_i[\sigma, u] = G^*_i[\sigma+1, u+1]$.

4.3.5 G^*_i is Sylvester-Like

Combining the two previous properties, the block G^*_i can be expressed as

$$G^*_i = \begin{bmatrix} X_{0,i+1} & & \\ \vdots & \ddots & \\ X_{m-1,i+1} & \cdots & X_{0,i+1} \\ X_{m,i+1} & \cdots & X_{1,i+1} \\ & \ddots & \vdots \\ & & X_{m,i+1} \end{bmatrix}.$$

4.4 Example: $m = 1$ and $n = 2$

$$M^*_{1,2} = \begin{bmatrix} -R_{0,1;0,0} & -R_{0,2;0,0} \\ -R_{1,1;0,0} + R_{1,0;0,1} & -R_{1,2;0,0} + R_{1,0;0,2} \\ -R_{1,1;1,0} & -R_{1,2;1,0} \\ -R_{0,2;0,0} & -R_{0,2;0,1} \\ -R_{1,2;0,0} + R_{1,0;0,2} & -R_{1,2;0,1} + R_{1,1;0,2} \\ -R_{1,2;1,0} & -R_{1,2;1,1} \end{bmatrix},$$

$$\begin{aligned} S^*_{1,2}G^*_{1,2} &= \begin{bmatrix} L_{0,0} & 0 & 0 & 0 \\ L_{1,0} & L_{0,0} & 0 & 0 \\ 0 & L_{1,0} & 0 & 0 \\ L_{0,1} & 0 & L_{0,0} & 0 \\ L_{1,1} & L_{0,1} & L_{1,0} & L_{0,0} \\ 0 & L_{1,1} & 0 & L_{1,0} \\ L_{0,2} & 0 & L_{0,1} & 0 \\ L_{1,2} & L_{0,2} & L_{1,1} & L_{0,1} \\ 0 & L_{1,2} & 0 & L_{1,1} \\ 0 & 0 & L_{0,2} & 0 \\ 0 & 0 & L_{1,2} & L_{0,2} \\ 0 & 0 & 0 & L_{1,2} \end{bmatrix} \begin{bmatrix} X_{0,1} & X_{0,2} \\ X_{1,1} & X_{1,2} \\ X_{0,2} & 0 \\ X_{1,2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -R_{0,1;0,0} & -R_{0,2;0,0} \\ -R_{1,1;0,0} + R_{1,0;0,1} & -R_{1,2;0,0} + R_{1,0;0,2} \\ -R_{1,1;1,0} & -R_{1,2;1,0} \\ -R_{0,2;0,0} & -R_{0,2;0,1} \\ -R_{1,2;0,0} + R_{1,0;0,2} & -R_{1,2;0,1} + R_{1,1;0,2} \\ -R_{1,2;1,0} & -R_{1,2;1,1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

5 From Sylvester to Cayley

In this section we revert to the default lexicographic orders $s > t$ and $\alpha > \beta$. We shall now derive a $6mn \times 2mn$ conversion matrix $F_{m,n}$ such that

$$S_{m,n}F_{m,n} = \begin{bmatrix} C_{m,n} \\ 0 \end{bmatrix}. \tag{13}$$

This result is very satisfying because it extends the following well-known expansion by cofactors property of 3×3 determinants

$$|i, j; k, l; p, q| = \begin{bmatrix} a_{i,j} & b_{i,j} & c_{i,j} \end{bmatrix} \begin{bmatrix} A_{k,l,p,q} \\ B_{k,l,p,q} \\ C_{k,l,p,q} \end{bmatrix} = L_{i,j}R^T_{k,l;p,q}.$$

The derivation below produces an explicit entry formula that reveals the simple elegant block structure of $F_{m,n}$.

5.1 The Conversion Matrix $F_{m,n}$

To find $F_{m,n}$, we expand the numerator on the right hand side of Equation (2) by cofactors of the first row to obtain

$$\Delta_{m,n} = \left(f(s,t) \frac{\left| \begin{array}{cc} g(\alpha,t) & h(\alpha,t) \\ g(\alpha,\beta) & h(\alpha,\beta) \end{array} \right|}{(\beta - t)} + \cdots \right) \frac{1}{(\alpha - s)}. \tag{14}$$

Expanding the 2×2 numerator determinant yields

$$\Delta_{m,n} = \left(f(s,t) \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=0}^{m} \sum_{q=0}^{n} \frac{A_{k,l;p,q} t^l \alpha^{k+p} \beta^q}{(\beta - t)} + \cdots \right) \frac{1}{(\alpha - s)}.$$

Applying the technique of truncated formal power series by regarding the numerator as a polynomial in β leads to

$$\Delta_{m,n} = \left(f(s,t) \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=0}^{m} \sum_{q=0}^{n} A_{k,l;p,q} \alpha^{k+p} \sum_{v=0}^{q-1} t^{l+q-1-v} \beta^v + \cdots \right) \frac{1}{(\alpha - s)}.$$

Rearranging the range of q and v yields

$$\Delta_{m,n} = \left(f(s,t) \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=0}^{m} \sum_{v=0}^{n-1} \sum_{q=v+1}^{n} A_{k,l;p,q} \alpha^{k+p} t^{l+q-1-v} \beta^v + \cdots \right) \frac{1}{(\alpha - s)}.$$

But the degree in t of the sum is at most $n-1$; hence the summation range of q can be made more precise:

$$\Delta_{m,n} = \left(f(s,t) \sum_{v=0}^{n-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=0}^{m} \sum_{q=v+1}^{\min(n,n+v-l)} A_{k,l;p,q} \alpha^{k+p} t^{l+q-1-v} \beta^v + \cdots \right) \frac{1}{(\alpha - s)}.$$

Again applying the technique of truncated formal power series by regarding the numerator as a polynomial in α leads to

$$\Delta_{m,n} = \sum_{v=0}^{n-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=0}^{m} \sum_{q=v+1}^{\min(n,n+v-l)} f(s,t) A_{k,l;p,q} t^{l+q-1-v} \beta^v \sum_{u=0}^{k+p-1} s^{k+p-1-u} \alpha^u + \cdots$$

Rearranging the range of p and u yields

$$\Delta_{m,n} = \sum_{u=0}^{2m-1} \sum_{v=0}^{n-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=\max(0,u+1-k)}^{m} \sum_{q=v+1}^{\min(n,n+v-l)} f(s,t) A_{k,l;p,q} s^{k+p-1-u} t^{l+q-1-v} \alpha^u \beta^v + \cdots.$$

Replacing the "$\cdots$" with the corresponding expressions for $g(s,t)$ and $h(s,t)$ produces

$$\Delta_{m,n} = \sum_{u=0}^{2m-1} \sum_{v=0}^{n-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=\max(0,u+1-k)}^{m} \sum_{q=v+1}^{\min(n,n+v-l)} (f A_{k,l;p,q} + g B_{k,l;p,q} + h C_{k,l;p,q}) s^{k+p-1-u} t^{l+q-1-v} \alpha^u \beta^v.$$

Writing $L = \left[\begin{array}{ccc} f & g & h \end{array} \right]$ leads to

$$\Delta_{m,n} = \sum_{u=0}^{2m-1} \sum_{v=0}^{n-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{p=\max(0,u+1-k)}^{m} \sum_{q=v+1}^{\min(n,n+v-l)} s^{k+p-1-u} t^{l+q-1-v} L R_{k,l;p,q}^T \alpha^u \beta^v. \tag{15}$$

Since $k \le m$, $u+1-k \le p \le m$, $u \ge 0$, $v+1 \le q \le n+v-l$, we have $0 \le k+p-1-u \le 2m-1$ and $0 \le l+q-1-v \le n-1$. Hence there is a $6mn \times 2mn$ matrix $F_{m,n}$ such that $\Delta_{m,n}$ can be written in matrix form as

$$\Delta_{m,n} = \begin{bmatrix} L^T \\ \vdots \\ s^\sigma t^\tau L^T \\ s^\sigma t^{\tau+1} L^T \\ \vdots \\ s^{2m-1} t^{n-1} L^T \end{bmatrix}^T F_{m,n} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{2m-1} \beta^{n-1} \end{bmatrix}. \tag{16}$$

It follows by Equation (16) and the definition of $S_{m,n}$ (Equation (1)) that

$$\Delta_{m,n} = \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{3m-1} t^{2n-1} \end{bmatrix}^T S_{m,n} F_{m,n} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{2m-1} \beta^{n-1} \end{bmatrix}. \tag{17}$$

Thus by the definition of $C_{m,n}$ (Equation (3)), we see that

$$S_{m,n} F_{m,n} = \begin{bmatrix} C_{m,n} \\ 0 \end{bmatrix}. \tag{18}$$

Notice that $4mn$ rows of zeros must be appended after the rows of $C_{m,n}$ because the row indices s^σ of $S_{m,n}$ run from 0 to $3m-1$ but the row indices s^σ of $C_{m,n}$ run from 0 to $m-1$.

5.2 The Entries of $F_{m,n}$

Solving $k+p-1-u=\sigma$ and $l+q-1-v=\tau$ in Equation (15) gives

$$\begin{aligned} p &= \sigma+1+u-k, \\ q &= \tau+1+v-l. \end{aligned}$$

The constraints

$$\begin{aligned} 0 \le\ & k \le m, \\ 0 \le\ & l \le n, \\ \max(0, u+1-k) \le\ & p \le m, \\ v+1 \le\ & q \le \min(n, n+v-l), \end{aligned}$$

lead to

$$\begin{aligned} \max(0, \sigma+1+u-m) \le\ & k \le \min(m, \sigma+1+u), \\ \max(0, \tau+1+v-n) \le\ & l \le \tau. \end{aligned}$$

Therefore the entry of the cofactor matrix $F_{m,n}$ indexed by $(s^\sigma t^\tau, \alpha^u \beta^v)$ is the sum

$$\sum_{k=\max(0,\sigma+1+u-m)}^{\min(m,\sigma+1+u)} \sum_{l=\max(0,\tau+1+v-n)}^{\tau} R^T_{k,l,\sigma+1+u-k,\tau+1+v-l}, \tag{19}$$

for $0 \le \sigma,\ u \le 2m-1,\ 0 \le \tau,\ v \le n-1$. But observe that the above sum contains mutually cancelling cofactors of the form $R^T_{k,l;p,q} + R^T_{p,q;k,l} = 0$ because $A_{k,l;p,q} + A_{p,q;k,l} = 0$, $B_{k,l;p,q} + B_{p,q;k,l} = 0$, $C_{k,l;p,q} + C_{p,q;k,l} = 0$. To find the cancelling partner of $R^T_{k,l;\sigma+1+u-k,\tau+1+v-l}$, let

$$\begin{aligned} k &= \sigma + 1 + u - k', \\ l &= \tau + 1 + v - l', \\ \sigma + 1 + u - k &= k', \\ \tau + 1 + v - l &= l'. \end{aligned}$$

A quick check reveals that when $v + 1 \le l$, we have

$$\begin{aligned} \max(0, \sigma + 1 + u - m) \le {} & k' \le \min(m, \sigma + 1 + u), \\ \max(0, \tau + 1 + v - n) \le {} & l' \le \tau. \end{aligned}$$

That is, the range of l need not run beyond v because terms out of that range mutually cancel. Consequently, the sum for the entry formula (19) of $F_{m,n}$ will be free of mutually cancelling terms $R^T_{k,l;p,q}$ if it is written as

$$\sum_{k=\max(0,\sigma+1+u-m)}^{\min(m,\sigma+1+u)} \ \sum_{l=\max(0,\tau+1+v-n)}^{\min(\tau,v)} R^T_{k,l;\sigma+1+u-k,\tau+1+v-l}, \tag{20}$$

for $0 \le \sigma,\ u \le 2m-1,\ 0 \le \tau,\ v \le n-1$.

5.3 Properties of $F_{m,n}$

The structure of $F_{m,n}$ is most revealing if we view this $6mn \times 2mn$ matrix as a square $2mn \times 2mn$ matrix by considering its entries as sums of the 3×1 matrices $R^T_{k,l;p,q}$.

5.3.1 $F_{m,n}$ is Symmetric

From the entry formula (20), we see that the entries indexed by $(s^\sigma t^\tau, \alpha^u \beta^v)$ and $(s^u t^v, \alpha^\sigma \beta^\tau)$ are the same. In other words, the matrix $F_{m,n}$ is symmetric.

5.3.2 $F_{m,n}$ is Block Upper Triangular

Let the $n \times n$ submatrix of $F_{m,n}$ indexed by $(s^\sigma t^\tau, \alpha^u \beta^v)$, $0 \le \tau,\ v \le n-1$, be $B_{\sigma,u}$. When $\sigma + u > 2m-1$ in the entry formula (20), we have

$$\sum_{k=\max(0,\sigma+u+1-m)}^{\min(m,\sigma+u+1)} = \sum_{k=\sigma+u+1-m}^{m}, \tag{21}$$

and clearly this summation range is vacuous. Thus the sum is zero. Consequently the conversion matrix $F_{m,n}$ is block upper triangular in the sense that the blocks $B_{\sigma,u}$ with $\sigma + u > 2m-1$ are all zero.

5.3.3 $F_{m,n}$ is Block Symmetric and Striped

From the entry formula (20), we easily see that

$$B_{\sigma-1,u+1} = B_{\sigma,u} = B_{\sigma+1,u-1}.$$

Therefore, the notation $F_{\sigma+u} = B_{\sigma,u}$ is well-defined, and

$$F_{m,n} = \begin{bmatrix} F_0 & F_1 & \cdots & F_{2m-1} \\ F_1 & \iddots & \iddots & \\ \iddots & \iddots & & \\ F_{2m-1} & & & \end{bmatrix}. \tag{22}$$

5.3.4 F_k's are Symmetric

Notice that τ and v are symmetric in formula (20); thus

$$B_{\sigma,u} \quad = \quad B^T_{\sigma,u}.$$

That is, each of the blocks F_k is symmetric.

5.3.5 Three Levels of Symmetry

In summary, $F_{m,n}$ has the following symmetry properties:

- $F_{m,n}$ is symmetric if we view $F_{m,n}$ as a $2mn \times 2mn$ square matrix, whose entries are 3×1 submatrices, i.e. sums of the $R^T_{k,l;p,q}$;
- $F_{m,n}$ is also symmetric if we view $F_{m,n}$ as a $2m \times 2m$ square matrix, whose entries are $3n \times n$ matrices, i.e. the F_k's;
- the submatrices F_k are symmetric if we view each F_k as an $n \times n$ matrix, whose entries are 3×1 submatrices, i.e. sums of the $R^T_{k,l;p,q}$.

5.3.6 F_k's are Bezoutian

Let

$$f_i(t) = \textstyle\sum_{j=0}^n a_{i,j}t^j, \qquad g_i(t) = \sum_{j=0}^n b_{i,j}t^j, \qquad h_i(t) = \sum_{j=0}^n c_{i,j}t^j.$$

Using the technique of truncated formal power series, we can write Equation (14) as

$$\begin{aligned}\Delta_{m,n} &= \left[f(s,t)\left(\sum_{i=0}^m g_i(t)\alpha^i \sum_{j=0}^m h_j(\beta)\alpha^j - \sum_{i=0}^m g_i(\beta)\alpha^i \sum_{j=0}^m h_j(t)\alpha^j\right)\frac{1}{\beta - t} + \cdots\right]\cdot\frac{1}{\alpha - s} \\ &= \sum_{i=0}^m\sum_{j=0}^m\sum_{u=0}^{i+j-1}\left(s^{i+j-1-u}\ \alpha^u\ f(s,t)\ \frac{g_i(t)h_j(\beta) - g_i(\beta)h_j(t)}{\beta - t} + \cdots\right)\end{aligned}$$

Therefore, the entries of the submatrix $B_{\sigma,u}$ ($= F_{\sigma+u}$) come from the coefficients of

$$\begin{aligned}&\sum_{i+j-1=\sigma+u}\frac{g_i(t)h_j(\beta) - g_i(\beta)h_j(t)}{\beta - t}, \\ &\sum_{i+j-1=\sigma+u}\frac{h_i(t)f_j(\beta) - h_i(\beta)f_j(t)}{\beta - t}, \\ &\sum_{i+j-1=\sigma+u}\frac{f_i(t)g_j(\beta) - f_i(\beta)g_j(t)}{\beta - t}.\end{aligned} \tag{23}$$

Each term in these three sums is a Cayley expression for two univariate polynomials $\{g_i, h_j\}$, $\{h_i, f_j\}$, and $\{f_i, g_j\}$; hence each term generates a Bezout matrix when written in matrix form [De Montaudouin and Tiller 1984; Goldman *et al* 1984; Chionh *et al* 1998]. Therefore, each block $F_{\sigma+u} = B_{\sigma,u}$ contains three summations of Bezout matrices, where the three matrices interleave row by row. In particular, if $\sigma + u = 2m - 1$, then $i = j = m$ in expression (23), so F_{2m-1} is the matrix obtained by interleaving the rows of the three Bezout matrices: Bezout(g_m, h_m), Bezout(h_m, f_m), Bezout(f_m, g_m).

Explicit formulas for the entries of the Bezout matrix of two univariate polynomials are given in [Goldman *et al* 1984]. More efficient methods for computing the entries of a Bezout resultant or the entries of a sum of Bezout resultants are described in [Chionh *et al* 1998]. Since the blocks F_k are Bezoutian, we can adopt these methods here to provide efficient algorithm for computing the entries of F_k. Alternatively, we can use Equation (20) directly together with the methods described in Section 7 to develop an efficient algorithm for computing the entries of F_k. Fast computation of the Bezoutiants F_k is important because, as we shall see in Section 7, we can use the blocks F_k to speed up the computation of the entries of the Cayley matrix $C_{m,n}$.

5.4 Example: $m = 1$ and $n = 2$

$$C_{1,2} = \left[\begin{array}{cc} |1,0;0,1;0,0| & |1,0;0,2;0,0| \\ |1,0;0,2;0,0| + |1,1;0,1;0,0| & |1,0;0,2;0,1| + |1,1;0,2;0,0| \\ |1,2;0,1;0,0| + |1,1;0,2;0,0| & |1,1;0,2;0,1| + |1,2;0,2;0,0| \\ |1,2;0,2;0,0| & |1,2;0,2;0,1| \end{array} \right.$$

$$\left. \begin{array}{cc} -|1,1;1,0;0,0| & -|1,2;1,0;0,0| \\ -|1,2;1,0;0,0| - |1,1;1,0;0,1| & -|1,2;1,0;0,1| - |1,2;1,1;0,0| \\ -|1,2;1,0;0,1| - |1,1;1,0;0,2| & -|1,2;1,1;0,1| - |1,2;1,0;0,2| \\ -|1,2;1,0;0,2| & -|1,2;1,1;0,2| \end{array} \right],$$

and $S_{1,2}F_{1,2} =$

$$\begin{bmatrix} L_{0,0} & 0 & 0 & 0 \\ L_{0,1} & L_{0,0} & 0 & 0 \\ L_{0,2} & L_{0,1} & 0 & 0 \\ 0 & L_{0,2} & 0 & 0 \\ L_{1,0} & 0 & L_{0,0} & 0 \\ L_{1,1} & L_{1,0} & L_{0,1} & L_{0,0} \\ L_{1,2} & L_{1,1} & L_{0,2} & L_{0,1} \\ 0 & L_{1,2} & 0 & L_{0,2} \\ 0 & 0 & L_{1,0} & 0 \\ 0 & 0 & L_{1,1} & L_{1,0} \\ 0 & 0 & L_{1,2} & L_{1,1} \\ 0 & 0 & 0 & L_{1,2} \end{bmatrix} \begin{bmatrix} -R^T_{1,1;0,0} + R^T_{1,0;0,1} & -R^T_{1,2;0,0} + R^T_{1,0;0,2} & -R^T_{1,1;1,0} & -R^T_{1,2;1,0} \\ -R^T_{1,2;0,0} + R^T_{1,0;0,2} & -R^T_{1,2;0,1} + R^T_{1,1;0,2} & -R^T_{1,2;1,0} & -R^T_{1,2;1,1} \\ -R^T_{1,1;1,0} & -R^T_{1,2;1,0} & 0 & 0 \\ -R^T_{1,2;1,0} & -R^T_{1,2;1,1} & 0 & 0 \end{bmatrix}.$$

Now it is straightforward to check that indeed

$$S_{1,2}F_{1,2} = \begin{bmatrix} C_{1,2} \\ 0 \end{bmatrix}.$$

6 From Mixed Cayley-Sylvester to Cayley

We have shown that the polynomials represented by the columns of the Cayley resultant and the mixed Cayley-Sylvester resultant are linear combinations of the polynomials represented by the columns of the Sylvester resultant. There is, however, no way to represent the polynomials represented by the columns of the Cayley resultant as linear combinations of the polynomials represented by the columns of the mixed

Cayley-Sylvester resultant because the Cayley columns represent polynomials of bidegree $(m-1, 2n-1)$ whereas the Cayley-Sylvester columns represent polynomials of bidegree $(3m-1, n-1)$. Thus while the Cayley matrix is smaller than the Cayley-Sylvester matrix, the polynomials in the column space of the Cayley matrix do not form a subspace of the polynomials in the column space of the Cayley-Sylvester matrix. On the other hand, by Equation (3), the rows of the Cayley matrix represent polynomials of bidegree $(2m-1, n-1)$. Therefore since the columns of the Cayley-Sylvester matrix are linearly independent, the polynomials represented by the rows of the Cayley matrix can be expressed as linear combinations of the polynomials represented by the columns of the Cayley-Sylvester matrix. Below we will derive the conversion matrix $E_{m,n}$ that transforms $M_{m,n}$ to $C^T_{m,n}$.

6.1 The Conversion Matrix $E_{m,n}$

To find $E_{m,n}$, we expand the determinant on the right hand side of Equation (2) with respect to the first row. Recall from Section 2.3 that

$$\begin{aligned}\begin{vmatrix} g(\alpha,t) & h(\alpha,t) \\ g(\alpha,\beta) & h(\alpha,\beta)\end{vmatrix} /(\beta-t) &= \begin{vmatrix} g(\alpha,\beta) & h(\alpha,\beta) \\ g(\alpha,t) & h(\alpha,t)\end{vmatrix} /(t-\beta) \\ &= \sum_{v=0}^{n-1} \overline{f}_v(\alpha,\beta)t^v.\end{aligned}$$

Therefore,

$$\Delta_{m,n}(s,t,\alpha,\beta) = -\left(f(s,t)\sum_{v=0}^{n-1} \overline{f}_v(\alpha,\beta)t^v + g(s,t)\sum_{v=0}^{n-1} \overline{g}_v(\alpha,\beta)t^v + h(s,t)\sum_{v=0}^{n-1} \overline{h}_v(\alpha,\beta)t^v \right) \frac{1}{s-\alpha}. \quad (24)$$

Using the notation of Section 2.3, write $\overline{L}_v = [\,\overline{f}_v \quad \overline{g}_v \quad \overline{h}_v\,]$. Since the numerator is divisible by the denominator in Equation (24), we can apply the technique of truncated formal power series by treating the numerator as a polynomial in s to write $\Delta_{m,n}(s,t,\alpha,\beta)$ as

$$\begin{aligned}\Delta_{m,n}(s,t,\alpha,\beta) &= -\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{\sigma=0}^{i-1}\sum_{v=0}^{n-1} \overline{L}_v \cdot \begin{bmatrix} a_{i,j} \\ b_{i,j} \\ c_{i,j}\end{bmatrix} \alpha^{i-1-\sigma}s^\sigma t^{j+v} \qquad (25) \\ &= [\,\overline{L}_0 \quad \cdots \quad \alpha^u\overline{L}_v \quad \alpha^u\overline{L}_{v+1} \quad \cdots \quad \alpha^{m-1}\overline{L}_{n-1}\,]\, E_{m,n} \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1}t^{2n-1}\end{bmatrix},\end{aligned}$$

for some coefficient matrix $E_{m,n}$. Hence by Equation (5),

$$\Delta_{m,n}(s,t,\alpha,\beta) = \begin{bmatrix} 1 \\ \vdots \\ \alpha^u\beta^v \\ \alpha^u\beta^{v+1} \\ \vdots \\ \alpha^{3m-1}\beta^{n-1}\end{bmatrix}^T M_{m,n}E_{m,n} \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1}t^{2n-1}\end{bmatrix}.$$

On the other hand, by Equation (3),

$$\begin{aligned}
\Delta_{m,n}(s,t,\alpha,\beta) &= \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1}t^{2n-1} \end{bmatrix}^T C_{m,n} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{2m-1}\beta^{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{2m-1}\beta^{n-1} \end{bmatrix}^T C_{m,n}^T \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1}t^{2n-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{3m-1}\beta^{n-1} \end{bmatrix}^T \begin{bmatrix} C_{m,n}^T \\ 0_{mn\times 2mn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1}t^{2n-1} \end{bmatrix}.
\end{aligned}$$

Therefore,

$$M_{m,n} \cdot E_{m,n} = \begin{bmatrix} C_{m,n}^T \\ 0_{mn\times 2mn} \end{bmatrix}. \tag{26}$$

6.2 The Entries of $E_{m,n}$

From Equation (25), the 3×1 submatrix of $E_{m,n}$ indexed by $(\alpha^{i-1-\sigma}\overline{L}_v, s^\sigma t^{j+v})$ is $-L_{i,j}^T$. By solving

$$\begin{aligned} u &= i-1-\sigma, \\ \tau &= j+v, \end{aligned}$$

in Equation (25), we get

$$\begin{aligned} i &= \sigma+u+1, \\ j &= \tau - v. \end{aligned}$$

Therefore the 3×1 submatrix of $E_{m,n}$ indexed by $(\alpha^u \overline{L}_v, s^\sigma t^\tau)$ is

$$-L_{\sigma+u+1,\tau-v}^T, \tag{27}$$

when $\sigma+u+1 \le m$, $0 \le \tau - v \le n$, and zero otherwise.

6.3 Properties of $E_{m,n}$

From the above entry formula (27), arguments similar to those of Section 4.3 or 5.3 show that $E_{m,n}$ has the following striped block symmetric structure:

$$E_{m,n} = \begin{bmatrix} E_0 & E_1 & \cdots & E_{m-1} \\ E_1 & \iddots & \iddots & \\ \vdots & \iddots & & \\ E_{m-1} & & & \end{bmatrix},$$

where

$$E_i = \begin{bmatrix} -L_{i+1,0}^T & -L_{i+1,1}^T & \cdots & -L_{i+1,n-1}^T & -L_{i+1,n}^T & 0 & \cdots & 0 \\ 0 & -L_{i+1,0}^T & \cdots & -L_{i+1,n-2}^T & -L_{i+1,n-1}^T & -L_{i+1,n}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -L_{i+1,0}^T & -L_{i+1,1}^T & -L_{i+1,2}^T & \cdots & -L_{i+1,n}^T \end{bmatrix}, \tag{28}$$

is of size $n \times 2n$ with each entry a 3×1 submatrix.

6.4 Example: $m = 1$ and $n = 2$

$$\begin{aligned} M_{1,2}E_{m,n} &= \begin{bmatrix} -R_{0,1,0,0} & -R_{0,2,0,0} \\ -R_{0,2,0,0} & -R_{0,2,0,1} \\ -R_{1,1,0,0} + R_{1,0,0,1} & -R_{1,2,0,0} + R_{1,0,0,2} \\ -R_{1,2,0,0} + R_{1,0,0,2} & R_{1,1,0,2} - R_{1,2,0,1} \\ -R_{1,1,1,0} & -R_{1,2,1,0} \\ -R_{1,2,1,0} & -R_{1,2,1,1} \end{bmatrix} \begin{bmatrix} -L_{1,0}^T & -L_{1,1}^T & -L_{1,2}^T & 0 \\ 0 & -L_{1,0}^T & -L_{1,1}^T & -L_{1,2}^T \end{bmatrix} \\ &= \begin{bmatrix} C_{1,2}^T \\ 0 \end{bmatrix}. \end{aligned}$$

7 The Block Structure of the Three Dixon Resultants

The natural block structures of the transformation matrices prompt us to reexamine once again the three Dixon resultant matrices to seek natural block structures compatible with the block structures of the transformation matrices. Below we shall:

1. describe these block structures;
2. analyze how these blocks are related to resultant matrices for univariate polynomials;
3. reveal how the blocks of the transformation matrices are related to the blocks of the resultant matrices;
4. simplify the calculation of the entries of the Cayley and mixed Cayley-Sylvester resultants by taking advantage of these block structures; and
5. derive identities relating the blocks of the Sylvester and the mixed Cayley-Sylvester resultants.

7.1 The Block Structure of the Sylvester Resultant

The block structure of $S_{m,n}$ is very natural and easy to describe. As in Section 5.3.6, we set

$$f_i(t) = \textstyle\sum_{j=0}^n a_{i,j}t^j, \qquad g_i(t) = \textstyle\sum_{j=0}^n b_{i,j}t^j, \qquad h_i(t) = \textstyle\sum_{j=0}^n c_{i,j}t^j.$$

Now let S_i be the $2n \times 3n$ coefficient matrix for the polynomials $(t^v f_i,\ t^v g_i,\ t^v h_i)$, $v = 0, \cdots, n-1$. Note that the matrix S_i is Sylvester-like in the sense that if we drop the f_i-columns from S_i, then we get the Sylvester matrix of g_i and h_i; dropping the g_i-columns yields the Sylvester matrix of h_i and f_i; dropping the h_i-columns yields the Sylvester matrix of f_i and g_i. Adopting the usual notation $L_{i,j} = [a_{i,j} \;\; b_{i,j} \;\; c_{i,j}]$, we can write

$$S_i = \begin{bmatrix} L_{i,0} & & \\ \vdots & \ddots & \\ L_{i,n-1} & \cdots & L_{i,0} \\ L_{i,n} & \cdots & L_{i,1} \\ & \ddots & \vdots \\ & & L_{i,n} \end{bmatrix}, \tag{29}$$

where the rows are indexed by the monomials $1, \cdots, t^{2n-1}$, and the columns are indexed by the polynomials $t^0[f_i \;\; g_i \;\; h_i], \cdots, t^{n-1}[f_i \;\; g_i \;\; h_i]$. It follows from Equation (1) that

$$S_{m,n} = \begin{bmatrix} S_0 & & & & \\ \vdots & \ddots & & & \\ S_{m-1} & \cdots & S_0 & & \\ S_m & \cdots & S_1 & S_0 & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & S_m & S_{m-1} & \cdots & S_0 \\ & & & S_m & \cdots & S_1 \\ & & & & \ddots & \vdots \\ & & & & & S_m \end{bmatrix}. \tag{30}$$

Notice from Equations (28) and (29) that

$$E_i = -(S_{i+1})^T, \qquad S_{i+1} = -(E_i)^T, \qquad 0 \le i \le m-1. \tag{31}$$

Similar results hold when we impose the order $t > s$. Define

$$f_j^*(s) = \textstyle\sum_{i=0}^m a_{i,j} s^i, \qquad g_j^*(s) = \sum_{i=0}^m b_{i,j} s^i, \qquad h_j^*(s) = \sum_{i=0}^m c_{i,j} s^i.$$

Now let S_j^* be the $3m \times 6m$ coefficient matrix for the polynomials $(s^v f_j^*, s^v g_j^*, s^v h_j^*)$, $v = 0, \cdots, 2m-1$. Then S_j^* can be written as

$$S_j^* = \begin{bmatrix} L_{0,j} & & & & \\ \vdots & \ddots & & & \\ L_{m-1,j} & \cdots & L_{0,j} & & \\ L_{m,j} & \cdots & L_{1,j} & L_{0,j} & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & L_{m,j} & L_{m-1,j} & \cdots & L_{0,j} \\ & & & L_{m,j} & \cdots & L_{1,j} \\ & & & & \ddots & \vdots \\ & & & & & L_{m,j} \end{bmatrix},$$

and it follows from Equation (1) (with the lexicographical order $t > s$) that

$$S_{m,n}^* = \begin{bmatrix} S_0^* & & \\ \vdots & \ddots & \\ S_{n-1}^* & \cdots & S_0^* \\ S_n^* & \cdots & S_1^* \\ & \ddots & \vdots \\ & & S_n^* \end{bmatrix}. \tag{32}$$

7.2 The Block Structure of the Mixed Cayley-Sylvester Resultant

The block structure of $M_{m,n}$ is also easy to describe, although the blocks themselves are more complicated. Recall from Section 2.3 that $\overline{L}_v = [\overline{f}_v \ \ \overline{g}_v \ \ \overline{h}_v]$, $v = 0, \cdots, n-1$, where $\overline{f}_v, \overline{g}_v, \overline{h}_v$ are polynomials of degree $2m$ in s and degree $n-1$ in t. Let M_i, $i = 0, \cdots, 3m-1$, be the $n \times 3n$ coefficient matrix of the monomials $s^i[1, \cdots, t^{n-1}]$ of the polynomials $\overline{L}_0, \cdots, \overline{L}_{n-1}$. Note that $M_i = 0$ when $i > 2m$ because every $\overline{L}_v$ is of degree $2m$ in s. Since the columns of $M_{m,n}$ are indexed by $s^i[\overline{L}_0, \cdots, \overline{L}_{n-1}]$, $i = 0, \cdots, m-1$, and the rows by $s^i[1, \cdots, t^{n-1}]$, $i = 0, \cdots, 3m-1$, it is easy to see from Equation (5) that $M_{m,n}$ has the following Sylvester-like (shifted block) structure:

$$M_{m,n} = \begin{bmatrix} M_0 & & \\ \vdots & \ddots & \\ M_{m-1} & \cdots & M_0 \\ \vdots & \vdots & \vdots \\ M_{2m} & \cdots & M_{m+1} \\ & \ddots & \vdots \\ & & M_{2m} \end{bmatrix}.$$

Using the notation f_i, g_i, h_i introduced in Section 5.3.6, we can derive explicit formulas for the blocks of $M_{m,n}$. To begin, write $\phi(g,h), \phi(h,f), \phi(f,g)$ as

$$\begin{aligned} \phi(g,h) &= \sum_{i=0}^{m}\sum_{j=0}^{m} \frac{g_i(t)h_j(\beta) - g_i(\beta)h_j(t)}{\beta - t}\, s^{i+j}, \\ \phi(h,f) &= \sum_{i=0}^{m}\sum_{j=0}^{m} \frac{h_i(t)f_j(\beta) - h_i(\beta)f_j(t)}{\beta - t}\, s^{i+j}, \\ \phi(f,g) &= \sum_{i=0}^{m}\sum_{j=0}^{m} \frac{f_i(t)g_j(\beta) - f_i(\beta)g_j(t)}{\beta - t}\, s^{i+j}. \end{aligned}$$

Now notice that the block M_k consists of the coefficients of s^k in these expressions. That is, M_k contains three matrices with their columns interleaved. Just like the expressions in Equation (23), the coefficients of s^k are sums of Cayley expressions. Hence each of these three matrices is the sum of the Bezout matrices of the polynomials $\{g_i, h_j\}$, $\{h_i, f_j\}$, $\{f_i, g_j\}$, for $i + j = k$. In particular, M_0 contains the three interleaved Bezout matrices: Bezout(g_0, h_0), Bezout(h_0, f_0), Bezout(f_0, g_0); similarly, M_{2m} contains three interleaved Bezout matrices: Bezout(g_m, h_m), Bezout(h_m, f_m), Bezout(f_m, g_m).

Notice that M_k and F_{k-1} [Section 5.3.6], $k = 1, \cdots, 2m$, contain the same three Bezoutian matrices; however, M_k is the $n \times 3n$ matrix where the three Bezoutian matrices interleave column by column, whereas F_{k-1} is the $3n \times n$ matrix where the three Bezoutian matrices interleave row by row. Since each Bezout matrix is symmetric [Goldman *et al* 1984], it is easy to see that

$$F_{k-1} = (M_k)^T, \qquad M_k = (F_{k-1})^T. \tag{33}$$

Therefore, we can apply the method of Section 5.3.6 to calculate the entries of either M_k or F_{k-1}, and obtain the entries of the other matrix without any additional computation.

The matrix $M^*_{m,n}$ has a different block structure. Recall from Equation (32) and Equation (12) that $S^*_{m,n}$ and $G^*_{m,n}$ can be written as

$$S^*_{m,n} = \begin{bmatrix} S^*_0 & & \\ \vdots & \ddots & \\ S^*_{n-1} & \cdots & S^*_0 \\ S^*_n & \cdots & S^*_1 \\ & \ddots & \vdots \\ & & S^*_n \end{bmatrix},$$

$$G^*_{m,n} = \begin{bmatrix} G^*_0 & \cdots & \cdots & G^*_{n-1} \\ \vdots & \iddots & \iddots & \\ \vdots & \iddots & & \\ G^*_{n-1} & & & \end{bmatrix},$$

where each S^*_i is of size $3m \times 6m$, and each G^*_j is of size $6m \times 3m$. But by Equation (10),

$$S^*_{m,n} \cdot G^*_{m,n} = \begin{bmatrix} M^*_{m,n} \\ 0 \end{bmatrix},$$

so

$$M^*_{m,n} = \begin{bmatrix} S^*_0 & & \\ \vdots & \ddots & \\ S^*_{n-1} & \cdots & S^*_0 \end{bmatrix} \cdot \begin{bmatrix} G^*_0 & \cdots & \cdots & G^*_{n-1} \\ \vdots & \iddots & \iddots & \\ \vdots & \iddots & & \\ G^*_{n-1} & & & \end{bmatrix}. \tag{34}$$

Write

$$M^*_{m,n} = \begin{bmatrix} K^*_{0,0} & \cdots & K^*_{0,n-1} \\ \vdots & \vdots & \vdots \\ K^*_{n-1,0} & \cdots & K^*_{n-1,n-1} \end{bmatrix},$$

where each block $K^*_{i,j}$ is of size $3m \times 3m$. By construction, each column of blocks

$$\begin{bmatrix} K^*_{0,j} \\ \vdots \\ K^*_{n-1,j} \end{bmatrix}$$

consists of the coefficients of the polynomials $[1, \cdots, s^{m-1}] \cdot \overline{L}_j$, $j = 0, \cdots, n-1$. Now by Equation (34),

$$K^*_{i,j} = \sum_{l=0}^{\min(i,n-1-j)} S^*_{i-l} \cdot G^*_{j+l}, \qquad 0 \le i,\ j \le n-1. \tag{35}$$

Notice, however, that Equation (35) involves a lot of redundant calculations along the diagonals $i+j=l$ for $l = 1, \cdots, 2n-3$. In fact,

$$K^*_{i,j} = S^*_i \cdot G^*_j + K^*_{i-1,j+1}. \tag{36}$$

To eliminate these redundancies, we can compute the entries of $M^*_{m,n}$ in the following way:

1. Initialization:

$$\left(M^*_{m,n}\right)_{init} = \begin{bmatrix} S^*_0 \cdot G^*_0 & \cdots & S^*_0 \cdot G^*_{n-1} \\ \vdots & \vdots & \vdots \\ S^*_{n-1} \cdot G^*_0 & \cdots & S^*_{n-1} \cdot G^*_{n-1} \end{bmatrix},$$

   that is,

$$\left(K^*_{i,j}\right)_{init} = S^*_i \cdot G^*_j, \qquad 0 \le i, j \le n-1.$$

2. Recursion:

   for $i = 1$ to $n-1$
   for $j = n-2$ to 0 (step -1)
   $K^*_{i,j} \leftarrow K^*_{i,j} + K^*_{i-1,j+1}$.

   That is, we add along the diagonals from upper right to lower left, so schematically,

$$M^*_{m,n} = \begin{bmatrix} S_0^* \cdot G_0^* & S_0^* \cdot G_1^* & \cdots & S_0^* \cdot G_{n-2}^* & S_0^* \cdot G_{n-1}^* \\ \swarrow & \swarrow & \iddots & \swarrow & S_1^* \cdot G_{n-1}^* \\ \iddots & \iddots & \iddots & \iddots & \vdots \\ \swarrow & \swarrow & \iddots & \swarrow & S_{n-2}^* \cdot G_{n-1}^* \\ \swarrow & \swarrow & \iddots & \swarrow & S_{n-1}^* \cdot G_{n-1}^* \end{bmatrix}.$$

The products of the lower $3mn$ rows of $S^*_{m,n}$ with $G^*_{m,n}$ generate the following n interesting identities:

$$\sum_{l=0}^{n-1-j} S^*_{n-l} \cdot G^*_{j+l} = 0, \qquad 0 \le j \le n-1. \tag{37}$$

Therefore by Equation (35) and Equation (37)

$$K^*_{n-1,j} = \sum_{l=0}^{n-1-j} S^*_{n-1-l} \cdot G^*_{j+l} = -S^*_n \cdot G^*_{j-1}, \qquad 1 \le j \le n-2,$$

so along the bottom row of blocks, we can initialize

$$K^*_{n-1,j} = -S^*_n \cdot G^*_{j-1}, \qquad j = 1, \cdots, n-2,$$

and remove the extraneous computations

$$K^*_{n-1,j} \leftarrow K^*_{n-1,j} + K^*_{n-2,j+1}, \qquad j = 1, \cdots, n-2.$$

7.3 The Block Structure of the Cayley Resultant

The block structure of $C_{m,n}$ is not so obvious as that of $S_{m,n}$ or $M_{m,n}$, since the entry formulas for $C_{m,n}$ are much more complicated [Chionh 1997]. However, informed by the transformation matrices $F_{m,n}$ and $E_{m,n}$ derived in Sections 5 and 6, we can impose an interesting block structure on $C_{m,n}$ and $C^T_{m,n}$.

7.3.1 The Block Structure of $C_{m,n}$

First recall from Equation (22) and Equation (33) that the conversion matrix $F_{m,n}$ can be written as

$$F_{m,n} = \begin{bmatrix} M_1^T & \cdots & \cdots & M_{2m}^T \\ \vdots & \iddots & \iddots & \\ \vdots & \iddots & & \\ M_{2m}^T & & & \end{bmatrix}, \tag{38}$$

where each block M_j^T is of size $3n \times n$. Since by Equation (18),

$$\begin{bmatrix} C_{m,n} \\ 0 \end{bmatrix} = S_{m,n} \cdot F_{m,n},$$

it follows from Equation (30) and Equation (38) that

$$C_{m,n} = \begin{bmatrix} S_0 & & & \\ S_1 & S_0 & & \\ \vdots & \ddots & \ddots & \\ S_{m-1} & S_{m-2} & \cdots & S_0 \end{bmatrix} \cdot \begin{bmatrix} M_1^T & \cdots & M_m^T & \cdots & M_{2m}^T \\ \vdots & \iddots & \iddots & \iddots & \\ M_m^T & \cdots & M_{2m}^T & & \end{bmatrix}. \tag{39}$$

Write

$$C_{m,n} = \begin{bmatrix} B_{0,0} & \cdots & B_{0,2m-1} \\ \vdots & \vdots & \vdots \\ B_{m-1,0} & \cdots & B_{m-1,2m-1} \end{bmatrix},$$

where each block $B_{i,j}$ is of size $2n \times n$. By construction, each column of blocks

$$\begin{bmatrix} B_{0,j} \\ \vdots \\ B_{m-1,j} \end{bmatrix},$$

$j = 0, \cdots, 2m-1$ consists of the coefficients of the polynomials in s, t that represent the coefficients of $\alpha^j[1, \cdots, \beta^{n-1}]$ in Cayley's determinant expression on the right hand side of Equation (2). Now by Equation (39),

$$B_{i,j} = \sum_{l=0}^{\min(i,2m-1-j)} S_{i-l} \cdot M_{j+l+1}^T, \qquad 0 \le i \le m-1, \;\; 0 \le j \le 2m-1. \tag{40}$$

Again, we observe that

$$B_{i,j} = B_{i-1,j+1} + S_i \cdot M_{j+1}^T, \qquad 1 \le i \le m-1, \; 0 \le j \le 2m-2.$$

which leads to a faster way for computing the entries of $C_{m,n}$:

1. Initialization:

$$(C_{m,n})_{init} = \begin{bmatrix} S_0 \cdot M_1^T & \cdots & S_0 \cdot M_m^T & S_0 \cdot M_{m+1}^T & \cdots & S_0 \cdot M_{2m}^T \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{m-1} \cdot M_1^T & \cdots & S_{m-1} \cdot M_m^T & S_{m-1} \cdot M_{m+1}^T & \cdots & S_{m-1} \cdot M_{2m}^T \end{bmatrix}.$$

   That is,

$$(B_{i,j})_{init} = S_i \cdot M_{j+1}^T, \qquad 0 \le i \le m-1, \;\; 0 \le j \le 2m-1.$$

2. Recursion:
   for $i = 1$ to $m-1$
   for $j = 2m-2$ to 0 (step -1)
   $B_{i,j} \leftarrow B_{i,j} + B_{i-1,j+1}$.

$$C_{m,n} = \begin{bmatrix} S_0 \cdot M_1^T & \cdots & S_0 \cdot M_m^T & S_0 \cdot M_{m+1}^T & \cdots & S_0 \cdot M_{2m}^T \\ \swarrow & \iddots & \swarrow & \swarrow & \iddots & S_1 \cdot M_{2m}^T \\ \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \swarrow & \iddots & \swarrow & \swarrow & \iddots & S_{m-2} \cdot M_{2m}^T \\ \swarrow & \iddots & \swarrow & \swarrow & \iddots & S_{m-1} \cdot M_{2m}^T \end{bmatrix}.$$

Notice that this method for computing the entries of $C_{m,n}$ eliminates lots of redundant calculations: after the initialization, we need only update the blocks along the southwest diagonals by one (block) addition per block. This approach is much faster than the standard method of calculating each entry independently [c.f. Sederberg 1983; Chionh 1997], especially when we compute the Bezoutiants M_k^T efficiently [c.f. Section 5.3.6]. Indeed this efficient computation of the entries of the Cayley resultant mirrors quite closely the efficient computation of the entries of the Bezout resultant [Chionh *et al* 1998].

Finally notice that the products of the lower $4mn$ rows of $S_{m,n}$ with $F_{m,n}$ generate the following $2m$ interesting identities:

$$\sum_{l=0}^{\min(m,2m-1-j)} S_{m-l} \cdot M_{j+l+1}^T = 0, \qquad 0 \le j \le 2m-1. \tag{41}$$

As in the previous section, these identities can be applied to simplify the computation of the last row of blocks of $C_{m,n}$ since by Equation (40) and Equation (41),

$$B_{m-1,j} = \sum_{l=0}^{\min(m-1,2m-1-j)} S_{m-1-l} \cdot M_{j+l+1}^T = -S_m \cdot M_j^T, \qquad 1 \le j \le 2m-2.$$

Thus we can initialize

$$B_{m-1,j} = -S_m \cdot M_j^T, \qquad 1 \le j \le 2m-2,$$

and remove the extraneous computation

$$B_{m-1,j} \leftarrow B_{m-1,j} + B_{m-2,j+1}, \qquad 1 \le j \le 2m-2.$$

7.3.2 The Block Structure of $C_{m,n}^T$ and the Convolution Identities

By Equation (26) and Equation (31), we can write $C_{m,n}^T$ as

$$C_{m,n}^T = \begin{bmatrix} M_0 & & \\ \vdots & \ddots & \\ M_{m-1} & \ddots & M_0 \\ M_m & \ddots & M_1 \\ & \ddots & \vdots \\ & & M_m \end{bmatrix} \cdot \begin{bmatrix} -S_1^T & \cdots & -S_m^T \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ -S_m^T & & \end{bmatrix}. \tag{42}$$

If $B_{i,j}$ is the $(i,j)^{\text{th}}$ block of $C_{m,n}$, then $(B_{j,i})^T$ is the $(i,j)^{\text{th}}$ block of $C_{m,n}^T$. Therefore, it follows from Equation (42) that

$$(B_{j,i})^T = -\sum_{l=0}^{\min(i,m-1-j)} M_{i-l} \cdot S_{j+1+l}^T, \quad 0 \le j \le m-1, 0 \le i \le 2m-1. \tag{43}$$

Hence

$$B_{i,j} = \sum_{l=0}^{\min(j,m-1-i)} S_{i+1+l} \cdot M_{j-l}^T, \quad 0 \le i \le m-1, 0 \le j \le 2m-1, \tag{44}$$

so

$$B_{i,j} = -S_{i+1} \cdot B_j^T + B_{i+1,j-1}. \tag{45}$$

Now using a method similar to that of Section 7.3.1, we can again compute the entries of $C_{m,n}$ very efficiently, except here after the initialization step we march to the northeast instead of to the southwest.

Combining the results in this section and the Section 7.3.1, we can derive the following remarkable convolution identities:

$$\sum_{l=\max(0,k-2m)}^{\min(k,m)} S_l \cdot M_{k-l}^T = 0, \quad 0 \le k \le 3m. \tag{46}$$

The identities in Equation (46) for $1 \le k \le 3m-1$ are obtained by comparing the two entry formulas Equation (40) and Equation (44); for $m+1 \le k \le 3m$, these identities are equivalent to Equation (41). Finally, the case of $k=0$ follows from $k=3m$. Observe that the matrices S_m and M_{2m} depend only on the polynomials f_m, g_m, h_m; in the same way the matrices S_0 and M_0 depend only on the polynomials f_0, g_0, h_0. Therefore, replacing the arbitrary polynomials f_m, g_m, h_m in the identity

$$S_m \cdot M_{2m}^T = 0 \quad (k = 3m),$$

by the polynomials f_0, g_0, h_0 yields the identity

$$S_0 \cdot M_0^T = 0 \quad (k = 0).$$

Parallel computation can be used to speed up the algorithms in this section and Section 7.2 even further. To begin, we can perform the initialization of the blocks in parallel. Moreover, the recursions along different diagonals are independent, so these steps can also be done in parallel. Finally, the identities in Equation (37) and Equation (41) can be used to simplify the computation of the last row of blocks in $C_{m,n}$ and $M^*_{m,n}$; similarly, Equation (44) can be applied to simplify the computation of the first column of blocks of $C_{m,n}$. Therefore the recursions can march simultaneously northeast and southwest if we choose the appropriate initialization.

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