Hybrid Dixon Resultants

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Abstract

Dixon [1908] describes three distinct homogeneous determinant representations for the resultant of three bivariate polynomials of bidegree (m, n). These Dixon resultants are the determinants of matrices of orders $6mn$, $3mn$ and $2mn$, and the entries of these matrices are respectively homogeneous of degrees 1, 2, and 3 in the coefficients of the original three polynomial equations. Here we mix and match columns from these three Dixon matrices to construct a large assortment of new hybrid determinant representations of orders ranging from $2mn$ to $6mn$ for the resultant of three bivariate polynomials of bidegree (m, n).

1 Introduction: Homogeneous and Hybrid Resultants

Free-form curves and surfaces in computer graphics and computer aided design are represented predominantly by polynomials or by rational functions based on Bezier, B-spline, and NURB formulations. Since resultants are an important mathematical tool for analyzing polynomial equations [Abhyankar 1976], it is not surprising that resultants have wide applications in the study and computation of computer generated curves and surfaces. Well-known practical applications include intersection [Kajiya 1982], implicitization [Sederberg *et al* 1984, Manocha and Canny 1992], and inversion [Chionh and Goldman 1992] of rational curves and surfaces. Theoretical applications also abound [Goldman *et al* 1984; Sederberg *et al* 1997; Cox *et al* 1998].

Resultants are often expressed as determinants. For two univariate polynomials of degree n, there are two standard determinant representations for the resultant: Sylvester's resultant [Van Der Waerden 1950] and Bezout's resultant [Goldman *et al* 1984; De Montaudouin and Tiller 1984]. The Sylvester resultant is the determinant of a $2n \times 2n$ matrix, whereas the Bezout resultant is the determinant of an $n \times n$ matrix. The Sylvester resultant is easier to construct, but the Bezout resultant is more compact, so it is easier to compute. Recently a new class of determinant formulations for the resultant of two univariate polynomials has been discovered, consisting of matrices of all orders between n (Bezout) and $2n$ (Sylvester). These new representations are hybrid matrices, containing k columns from the Bezout matrix and $2n-2k$ truncated columns from the Sylvester matrix [Weyman and Zelevinsky 1994; Sederberg *et al* 1997; Chionh *et al* 1998b]. When $k = 0$, these hybrids reduce to the usual Sylvester matrix, and when $k = n$, they reduce to the standard Bezout matrix.

We say that a determinant representation for a resultant is a *homogeneous resultant* if all the entries in the matrix are of the same degree in the coefficients of the generating polynomials. If the entries are of different degrees, we call the determinant representation for the resultant a *hybrid resultant*. The Sylvester and Bezout resultants for two univariate polynomials of degree n are homogeneous resultants, since the entries in the associated matrices are of degree one and two respectively in the coefficients of the original polynomials, but the new resultant matrices of orders between n and $2n$ are hybrid resultants because they contain some entries of degree one and some entries of degree two.

Hybrid resultants are also known for three polynomials of the same total degree in two variables. Sylvester [Salmon 1924] and Dixon [Dixon 1908] each describe hybrid methods for three polynomials of arbitrary common degree. When the common degree is two or three, a Jacobian method also applies [Muir 1933]. For a brief survey of these hybrid methods, see [Chionh and Goldman 1995].

For three bivariate polynomials of bidegree (m, n), Dixon describes three homogeneous determinant representations for the resultant [Dixon 1908]. The first formulation, generated by Sylvester's dialytic method, is the determinant of a $6mn \times 6mn$ matrix, whose entries are homogeneous of degree one in the coefficients of the original three polynomials. The second expression, constructed using an extension of Cayley's determinant device for generating the Bezout resultant for two univariate polynomials, is the determinant of a $2mn \times 2mn$ matrix, whose entries are homogeneous of degree three in the coefficients of the original three polynomials. The third representation, generated by combining Cayley's determinant device with Sylvester's dialytic method, is the determinant of a $3mn \times 3mn$ matrix, whose entries are homogeneous of degree two in the coefficients of the original three polynomials.

Dixon also presents a fourth, hybrid determinant representation for the resultant of three bivariate polynomials of bidegree (m, n), consisting of $3mn$ columns whose entries are homogeneous of degree 1 and mn columns whose entries are homogeneous of degree three [Dixon 1908]. This $4mn \times 4mn$ matrix is a hybrid of the Sylvester and Cayley constructions. The $3mn$ columns of degree 1 are generated by Sylvester's dialytic method, while the mn columns of degree three are generated using a variant of Cayley's determinant device.

Given the historical experience with the theoretical and computational differences between mathematically equivalent but representationally distinct resultant formulations, we set out to discover some new resultant expressions related to the Dixon resultants. The purpose of this paper is to construct a wide assortment of hybrid resultants for three bivariate polynomials of bidegree (m, n). In particular, we shall show that there are hybrid resultants of the following three general types:

1. hybrid Sylvester/Cayley resultants consisting of $6mn - 3kn$ truncated Sylvester columns and kn Cayley columns, $k = 0, \cdots, 2m$;
2. hybrid Sylvester/mixed Cayley-Sylvester resultants consisting of $6mn - 6km$ truncated Sylvester columns and $3km$ mixed Cayley-Sylvester columns, $k = 0, \cdots, n$;
3. hybrid mixed Cayley-Sylvester/Cayley resultants consisting of $3mn - 3kn$ truncated mixed Cayley-Sylvester columns and $2kn$ Cayley columns, $k = 0, \cdots, m$.

To derive these hybrid resultants, we shall need to exploit the block structures of the three homogeneous Dixon resultants along with the associated block structures of the accompanying matrices that transform between these three homogeneous representations for the resultant [Chionh *et al* 1998a]. We begin then with a review of the structure of the three homogeneous Dixon resultants and their associated transformation matrices.

2 The Three Dixon Resultants

Consider three bivariate polynomials of bidegree (m, n):

$$f(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j} s^i t^j, \qquad g(s,t) = \sum_{k=0}^{m}\sum_{l=0}^{n} b_{k,l} s^k t^l, \qquad h(s,t) = \sum_{p=0}^{m}\sum_{q=0}^{n} c_{p,q} s^p t^q.$$

Dixon describes three methods for constructing a homogenous resultant for f, g, h [Dixon 1908; Chionh *et al* 1998a]. In this section, we will outline the construction of these three Dixon resultants: the $6mn \times 6mn$ Sylvester resultant, the $3mn \times 3mn$ mixed Cayley-Sylvester resultant, and the $2mn \times 2mn$ Cayley resultant.

2.1 The Sylvester Resultant

Let $L = \begin{bmatrix} f & g & h \end{bmatrix}$. The determinant of the coefficient matrix $S_{m,n}$ of the $6mn$ polynomials $s^\sigma t^\tau L$, $0 \le \sigma \le 2m-1$, $0 \le \tau \le n-1$, is known as the *Sylvester resultant.*

To fix the order of the rows and columns of the Sylvester matrix, we impose either the lexicographical order $s > t$ so that

$$\begin{bmatrix} L & \cdots & s^\sigma t^\tau L & s^\sigma t^{\tau+1} L & \cdots & s^{2m-1} t^{n-1} L \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{3m-1} t^{2n-1} \end{bmatrix}^T S_{m,n}^{s>t}, \tag{1}$$

or the lexicographical order $t > s$ so that

$$\begin{bmatrix} L & \cdots & t^\tau s^\sigma L & t^\tau s^{\sigma+1} L & \cdots & t^{n-1} s^{2m-1} L \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ t^\tau s^\sigma \\ t^\tau s^{\sigma+1} \\ \vdots \\ t^{2n-1} s^{3m-1} \end{bmatrix}^T S_{m,n}^{t>s}. \tag{2}$$

Depending on the order we choose, the Sylvester resultant for f, g, h is either $|S_{m,n}^{s>t}|$ or $|S_{m,n}^{t>s}|$, which are the same up to sign.

Let

$$f_i(t) = \sum_{j=0}^{n} a_{i,j} t^j, \qquad g_i(t) = \sum_{j=0}^{n} b_{i,j} t^j, \qquad h_i(t) = \sum_{j=0}^{n} c_{i,j} t^j, \tag{3}$$

and let S_i be the $2n \times 3n$ coefficient matrix of the univariate polynomials $(t^v f_i,\ t^v g_i,\ t^v h_i)$, $v = 0, \cdots, n-1$. Then $S_{m,n}^{s>t}$ can be written as [Chionh *et al* 1998a]

$$S_{m,n}^{s>t} = \begin{bmatrix} S_0 & & & & & \\ \vdots & \ddots & & & & \\ S_{m-1} & \cdots & S_0 & & & \\ S_m & \cdots & S_1 & S_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & S_m & S_{m-1} & \cdots & S_0 \\ & & & S_m & \cdots & S_1 \\ & & & & \ddots & \vdots \\ & & & & & S_m \end{bmatrix}. \tag{4}$$

Note that each block S_i is Sylvester-like in the sense that if we drop the f_i-columns, then we get the Sylvester matrix of the univariate polynomials g_i, h_i; dropping the g_i-columns yields the Sylvester matrix of h_i, f_i; dropping the h_i-columns yields the Sylvester matrix of f_i, g_i.

Similarly, let

$$f_j^*(s) = \sum_{i=0}^{m} a_{i,j} s^i, \qquad g_j^*(s) = \sum_{i=0}^{m} b_{i,j} s^i, \qquad h_j^*(s) = \sum_{i=0}^{m} c_{i,j} s^i, \tag{5}$$

and let S_j^* be the $3m \times 6m$ coefficient matrix of the univariate polynomials $(s^v f_j^*, s^v g_j^*, s^v h_j^*)$, $v = 0, \cdots, 2m-1$. Then $S_{m,n}^{t>s}$ can be written as [Chionh *et al* 1998a]

$$S_{m,n}^{t>s} = \begin{bmatrix} S_0^* & & \\ \vdots & \ddots & \\ S_{n-1}^* & \cdots & S_0^* \\ S_n^* & \cdots & S_1^* \\ & \ddots & \vdots \\ & & S_n^* \end{bmatrix}. \tag{6}$$

2.2 The Mixed Cayley-Sylvester Resultant

Let

$$\begin{aligned}
\phi(g,h) &= \begin{vmatrix} g(s,t) & h(s,t) \\ g(s,\beta) & h(s,\beta) \end{vmatrix} / (\beta - t) = \sum_{v=0}^{n-1} \overline{f}_v(s,t)\beta^v, \\
\phi(h,f) &= \begin{vmatrix} h(s,t) & f(s,t) \\ h(s,\beta) & f(s,\beta) \end{vmatrix} / (\beta - t) = \sum_{v=0}^{n-1} \overline{g}_v(s,t)\beta^v, \\
\phi(f,g) &= \begin{vmatrix} f(s,t) & g(s,t) \\ f(s,\beta) & g(s,\beta) \end{vmatrix} / (\beta - t) = \sum_{v=0}^{n-1} \overline{h}_v(s,t)\beta^v,
\end{aligned}$$

and let $\overline{L}_v = \begin{bmatrix} \overline{f}_v & \overline{g}_v & \overline{h}_v \end{bmatrix}$. The determinant of the coefficient matrix $M_{m,n}$ of the polynomial

$$\sum_{u=0}^{m-1} s^u \alpha^u \sum_{v=0}^{n-1} \overline{L}_v \beta^v$$

is known as the *mixed Cayley-Sylvester resultant.*

To fix the order of the rows and columns of the mixed Cayley-Sylvester matrix, we impose either the lexicographical orders $s > \overline{L}$, $s > t$ and $\alpha > \beta$ so that

$$\begin{aligned}
\sum_{u=0}^{m-1} s^u \alpha^u \sum_{v=0}^{n-1} \overline{L}_v \beta^v &= \begin{bmatrix} \overline{L}_0 & \cdots & \overline{L}_v s^\sigma & \overline{L}_{v+1} s^\sigma & \cdots & \overline{L}_{n-1} s^{m-1} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{m-1}\beta^{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{3m-1} t^{n-1} \end{bmatrix}^T M_{m,n}^{s>t} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{m-1}\beta^{n-1} \end{bmatrix},
\end{aligned} \tag{7}$$

or the orders $\overline{L} > s$, $t > s$, and $\beta > \alpha$ so that

$$\sum_{u=0}^{m-1} s^u \alpha^u \sum_{v=0}^{n-1} \overline{L}_v \beta^v = \begin{bmatrix} \overline{L}_0 & \cdots & \overline{L}_v s^\sigma & \overline{L}_v s^{\sigma+1} & \cdots & \overline{L}_{n-1} s^{m-1} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \beta^v \alpha^u \\ \beta^v \alpha^{u+1} \\ \vdots \\ \beta^{n-1}\alpha^{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \vdots \\ t^\tau s^\sigma \\ t^\tau s^{\sigma+1} \\ \vdots \\ t^{n-1} s^{3m-1} \end{bmatrix}^T M_{m,n}^{t>s} \begin{bmatrix} 1 \\ \vdots \\ \beta^v \alpha^u \\ \beta^v \alpha^{u+1} \\ \vdots \\ \beta^{n-1}\alpha^{m-1} \end{bmatrix}. \tag{8}$$

Again depending on which ordering we choose, the mixed Cayley-Sylvester resultant of f, g, h is $|M_{m,n}^{s>t}|$ or $|M_{m,n}^{t>s}|$. Note that in each case, $M_{m,n}$ is a square matrix of order $3mn$, though in both cases we write $M_{m,n}$ as a matrix of size $3mn \times mn$ whose entries are 1×3 matrices.

Let M_i, $i = 0, \cdots, 3m-1$, be the $n \times 3n$ coefficient matrix of the monomials $s^i[1, \cdots, t^{n-1}]$ of the polynomials $\overline{L}_0, \cdots, \overline{L}_{n-1}$. Since each $\overline{L}_v$ is of degree $2m$ in s, $M_i = 0$ when $i > 2m$. Thus $M_{m,n}^{s>t}$ can be written as [Chionh *et al* 1998a]

$$M_{m,n}^{s>t} = \begin{bmatrix} M_0 & & \\ \vdots & \ddots & \\ M_{m-1} & \ddots & M_0 \\ \vdots & \ddots & \vdots \\ M_{2m} & \ddots & M_{m+1} \\ & \ddots & \vdots \\ & & M_{2m} \end{bmatrix}. \tag{9}$$

Moreover, M_{2m} is a Bezoutiant [Chionh *et al* 1998a]. Indeed, M_{2m} is a mixture of the three Bezout matrices for the polynomials $\{g_m, h_m\}$, $\{h_m, f_m\}$, $\{f_m, g_m\}$, where the three matrices interleave column by column, and f_m, g_m, h_m are univariate polynomials as defined in Equation (3). In other words, if we group together the first column out of every three columns of M_{2m}, we get the Bezout matrix of g_m, h_m; grouping together the second column out of every three columns yields the Bezout matrix of h_m, f_m; grouping together the third column out of every three columns yields the Bezout matrix of f_m, g_m. A simple formula for the entries of the other blocks M_k, $k = 0, \cdots, 2m-1$, is provided in [Chionh *et al* 1998a].

The block structure of $M_{m,n}^{t>s}$ is different from that of $M_{m,n}^{s>t}$. We write $M_{m,n}^{t>s}$ as

$$M_{m,n}^{t>s} = \begin{bmatrix} K_{0,0}^* & \cdots & K_{0,n-1}^* \\ \vdots & \vdots & \vdots \\ K_{n-1,0}^* & \cdots & K_{n-1,n-1}^* \end{bmatrix}, \tag{10}$$

where each block $K_{i,j}^*$ is of size $3m \times 3m$. In this block structure, each column of blocks

$$\begin{bmatrix} K_{0,j}^* \\ \vdots \\ K_{n-1,j}^* \end{bmatrix}$$

consists of the coefficients of the polynomials $[1, \cdots, s^{m-1}] \cdot \overline{L}_j$, $j = 0, \cdots, n-1$. A simple algorithm for computing the entries of each block $K^*_{i,j}$ is presented in [Chionh *et al* 1998a].

2.3 The Cayley Resultant

The determinant of the coefficient matrix of the Cayley expression

$$\Delta_{m,n}(s,t,\alpha,\beta) = \frac{\begin{vmatrix} f(s,t) & g(s,t) & h(s,t) \\ f(\alpha,t) & g(\alpha,t) & h(\alpha,t) \\ f(\alpha,\beta) & g(\alpha,\beta) & h(\alpha,\beta) \end{vmatrix}}{(\alpha - s)(\beta - t)} \tag{11}$$

indexed by $(s^\sigma t^\tau, \alpha^u \beta^v)$ is known as the *Cayley resultant.* To fix the order of the rows and columns of the Cayley matrix, we impose the lexicographical orders $s > t$ and $\alpha > \beta$ so that

$$\Delta_{m,n}(s,t,\alpha,\beta) = \begin{bmatrix} 1 \\ \vdots \\ s^\sigma t^\tau \\ s^\sigma t^{\tau+1} \\ \vdots \\ s^{m-1} t^{2n-1} \end{bmatrix}^T C_{m,n} \begin{bmatrix} 1 \\ \vdots \\ \alpha^u \beta^v \\ \alpha^u \beta^{v+1} \\ \vdots \\ \alpha^{2m-1} \beta^{n-1} \end{bmatrix}. \tag{12}$$

The Cayley resultant of f, g, h is $|C_{m,n}|$.

Similar to the block structure of $M^{t>s}_{m,n}$, $C_{m,n}$ can be written in the following way:

$$C_{m,n} = \begin{bmatrix} B_{0,0} & \cdots & B_{0,2m-1} \\ \vdots & \vdots & \vdots \\ B_{m-1,0} & \cdots & B_{m-1,2m-1} \end{bmatrix}, \tag{13}$$

where each block $B_{i,j}$ is of size $2n \times n$. The jth column of blocks

$$\begin{bmatrix} B_{0,j-1} \\ \vdots \\ B_{m-1,j-1} \end{bmatrix},$$

$j = 1, \cdots, 2m$, consists of the coefficients of the polynomials that are the coefficients of $\alpha^{j-1}[1, \cdots, \beta^{n-1}]$ in Cayley's determinant expression on the right hand side of Equation (11). Explicit formulas for the entries of $C_{m,n}$ are provided in [Chionh 1997]; a more efficient algorithm for calculating the entries of each block $B_{i,j}$ is presented in [Chionh *et al* 1998a].

3 Main Results: Hybrid Resultants

We outline our main results on hybrid resultants below, before we go into the details of the proofs.

3.1 Hybrids of $S^{s>t}_{m,n}$ and $C_{m,n}$

First we describe hybrids of $S^{s>t}_{m,n}$ and $C_{m,n}$. Recall that in Equation (4), $S^{s>t}_{m,n}$ has $2m$ columns of blocks S_i; in Equation (13), $C_{m,n}$ also has $2m$ columns of blocks $B_{i,j}$. We cumulatively replace the column of blocks of $S^{s>t}_{m,n}$ with index $(2m-1-j)$ by the column of blocks of $C_{m,n}$ with index j, $j = 0, \cdots, 2m-1$. The $j-$th hybrid matrix H_j has the following appearance:

$$H_j = \begin{bmatrix} S_0 & & & & B_{0,0} & \cdots & B_{0,j-1} \\ \vdots & \ddots & & & \vdots & \vdots & \vdots \\ S_{m-1} & & \ddots & & B_{m-1,0} & \cdots & B_{m-1,j-1} \\ S_m & & & \ddots & & & \\ & \ddots & & S_0 & & & \\ & & \ddots & \vdots & & & \\ & & & \ddots \; S_{m-1} & & & \\ & & & S_m & & & \end{bmatrix},$$

where the left part of H_j consists of the first $2m-j$ (not replaced) columns of $S^{s>t}_{m,n}$, and the right part consists of the first j (replacing) columns of $C_{m,n}$. In this way, we obtain a sequence of $2m-1$ hybrid matrices (the $2m-$th matrix is $C_{m,n}$), and we will show in Section 5 that, up to sign, the determinant of each hybrid matrix H_j is equal to $|S^{s>t}_{m,n}|$ (or $|C_{m,n}|$).

Example: $m = 2$ and $n = 3$.

$$H_0 = S^{s>t}_{2,3} = \begin{bmatrix} S_0 & 0 & 0 & 0 \\ S_1 & S_0 & 0 & 0 \\ S_2 & S_1 & S_0 & 0 \\ 0 & S_2 & S_1 & S_0 \\ 0 & 0 & S_2 & S_1 \\ 0 & 0 & 0 & S_2 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} S_0 & 0 & 0 & B_{0,0} \\ S_1 & S_0 & 0 & B_{1,0} \\ S_2 & S_1 & S_0 & 0 \\ 0 & S_2 & S_1 & 0 \\ 0 & 0 & S_2 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} S_0 & 0 & B_{0,0} & B_{0,1} \\ S_1 & S_0 & B_{1,0} & B_{1,1} \\ S_2 & S_1 & 0 & 0 \\ 0 & S_2 & 0 & 0 \end{bmatrix}, \qquad H_3 = \begin{bmatrix} S_0 & B_{0,0} & B_{0,1} & B_{0,2} \\ S_1 & B_{1,0} & B_{1,1} & B_{1,2} \\ S_2 & 0 & 0 & 0 \end{bmatrix},$$

$$H_4 = C_{2,3} = \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} \\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,3} \end{bmatrix},$$

where each S_i is of size 6×9, and each $B_{i,j}$ is of size 6×3.

3.2 Hybrids of $S^{t>s}_{m,n}$ and $M^{t>s}_{m,n}$

Next we will describe hybrids of $S^{t>s}_{m,n}$ and $M^{t>s}_{m,n}$. Notice that by Equations (6) and (10), $S^{t>s}_{m,n}$ and $M^{t>s}_{m,n}$ both have n columns of blocks. We cumulatively replace the column of blocks of $S^{t>s}_{m,n}$ with index $(n-1-j)$ by the column of blocks of $M^{t>s}_{m,n}$ with index j, $j = 0, \cdots, n-1$. In this case, the $j-$th hybrid is given by

$$H_j = \begin{bmatrix} S^*_0 & & & & K^*_{0,0} & \cdots & K^*_{0,j-1} \\ \vdots & \ddots & & & \vdots & \vdots & \vdots \\ S^*_{n-1} & & \ddots & & K^*_{n-1,0} & \cdots & K^*_{n-1,j-1} \\ S^*_n & & & \ddots & & & \\ & \ddots & & S^*_0 & & & \\ & & \ddots & \vdots & & & \\ & & & \ddots \; S^*_{n-1} & & & \\ & & & S^*_n & & & \end{bmatrix},$$

where the left part of H_j consists of the first $n-j$ columns of $S^{t>s}_{m,n}$ and the right part consists of the first j columns of $M^{t>s}_{m,n}$. In this way, we obtain a series of $n-1$ hybrid matrices (the n−th matrix is $M^{t>s}_{m,n}$), and we will show in Section 6 that, up to sign, the determinant of each hybrid H_j is the same as $|S^{t>s}_{m,n}|$ (or $|M^{t>s}_{m,n}|$).

Example: $m = 2$ and $n = 3$.

$$H_0 = S^{t>s}_{2,3} = \begin{bmatrix} S^*_0 & 0 & 0 \\ S^*_1 & S^*_0 & 0 \\ S^*_2 & S^*_1 & S^*_0 \\ S^*_3 & S^*_2 & S^*_1 \\ 0 & S^*_3 & S^*_2 \\ 0 & 0 & S^*_3 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} S^*_0 & 0 & K^*_{0,0} \\ S^*_1 & S^*_0 & K^*_{1,0} \\ S^*_2 & S^*_1 & K^*_{2,0} \\ S^*_3 & S^*_2 & 0 \\ 0 & S^*_3 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} S^*_0 & K^*_{0,0} & K^*_{0,1} \\ S^*_1 & K^*_{1,0} & K^*_{1,1} \\ S^*_2 & K^*_{2,0} & K^*_{2,1} \\ S^*_3 & 0 & 0 \end{bmatrix}, \qquad H_3 = M^{t>s}_{2,3} = \begin{bmatrix} K^*_{0,0} & K^*_{0,1} & K^*_{0,2} \\ K^*_{1,0} & K^*_{1,1} & K^*_{1,2} \\ K^*_{2,0} & K^*_{2,1} & K^*_{2,2} \end{bmatrix},$$

where each S^*_i is of size 6×12, and each $K^*_{i,j}$ is of size 6×6.

3.3 Hybrids of $M^{s>t}_{m,n}$ and $C^T_{m,n}$

Finally we describe hybrids of $M^{s>t}_{m,n}$ and $C^T_{m,n}$. From Equations (9) and (13) we see that the matrices $M^{s>t}_{m,n}$ and $C^T_{m,n}$ both have m columns of blocks. We replace the column of blocks of $M^{s>t}_{m,n}$ with index $(m-1-j)$ by the column of blocks of $C^T_{m,n}$ with index j, $j = 0, \cdots, m-1$. Now the j−th hybrid is given by

$$H_j = \begin{bmatrix} M_0 & & & & B^T_{0,0} & \cdots & B^T_{j-1,0} \\ \vdots & \ddots & & & \vdots & \vdots & \vdots \\ M_{2m-1} & & \ddots & & B^T_{0,2m-1} & \cdots & B^T_{j-1,2m-1} \\ M_{2m} & & & \ddots & & & \\ & \ddots & & M_0 & & & \\ & & \ddots & \vdots & & & \\ & & & \ddots \; M_{2m-1} & & & \\ & & & M_{2m} & & & \end{bmatrix},$$

where the left part of H_j contains the first $m-j$ columns of $M^{s>t}_{m,n}$ and right part contains the first j columns of $C^T_{m,n}$. In this way, we obtain $m-1$ hybrid matrices (the m−th matrix is $C^T_{m,n}$), and we will show in Section 7 that, up to sign, the determinant of each hybrid H_j is equal to $|M^{s>t}_{m,n}|$ (or $|C^T_{m,n}|$).

Example: $m = 3$ and $n = 3$.

$$H_0 = M^{s>t}_{3,3} = \begin{bmatrix} M_0 & 0 & 0 \\ M_1 & M_0 & 0 \\ M_2 & M_1 & M_0 \\ M_3 & M_2 & M_1 \\ M_4 & M_3 & M_2 \\ M_5 & M_4 & M_3 \\ M_6 & M_5 & M_4 \\ 0 & M_6 & M_5 \\ 0 & 0 & M_6 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} M_0 & 0 & B^T_{0,0} \\ M_1 & M_0 & B^T_{0,1} \\ M_2 & M_1 & B^T_{0,2} \\ M_3 & M_2 & B^T_{0,3} \\ M_4 & M_3 & B^T_{0,4} \\ M_5 & M_4 & B^T_{0,5} \\ M_6 & M_5 & 0 \\ 0 & M_6 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} M_0 & B_{0,0}^T & B_{1,0}^T \\ M_1 & B_{0,1}^T & B_{1,1}^T \\ M_2 & B_{0,2}^T & B_{1,2}^T \\ M_3 & B_{0,3}^T & B_{1,3}^T \\ M_4 & B_{0,4}^T & B_{1,4}^T \\ M_5 & B_{0,5}^T & B_{1,5}^T \\ M_6 & 0 & 0 \end{bmatrix}, \qquad H_3 = C_{3,3}^T = \begin{bmatrix} B_{0,0}^T & B_{1,0}^T & B_{2,0}^T \\ B_{0,1}^T & B_{1,1}^T & B_{2,1}^T \\ B_{0,2}^T & B_{1,2}^T & B_{2,2}^T \\ B_{0,3}^T & B_{1,3}^T & B_{2,3}^T \\ B_{0,4}^T & B_{1,4}^T & B_{2,4}^T \\ B_{0,5}^T & B_{1,5}^T & B_{2,5}^T \end{bmatrix},$$

where each M_i is of size 3×9, and each $B_{i,j}^T$ is of size 3×6.

4 Transformation Matrices and Their Properties

It is shown in [Chionh *et al* 1998a] that by multiplying to the right by a transformation matrix, one Dixon resultant matrix can be converted to another Dixon resultant matrix of smaller dimension. This section describes the block structures of these transformation matrices, which are essential for the derivation of the hybrid matrices in later sections. For further details and derivations, see [Chionh *et al* 1998a].

4.1 From Sylvester to Mixed Cayley-Sylvester

There is a $6mn \times 3mn$ matrix $G_{m,n}$ such that

$$S_{m,n}^{t>s} G_{m,n} = \begin{bmatrix} M_{m,n}^{t>s} \\ 0 \end{bmatrix}. \tag{14}$$

The matrix $G_{m,n}$ has the following block structure:

$$G_{m,n} = \begin{bmatrix} G_0 & G_1 & \cdots & G_{n-1} \\ G_1 & \iddots & \iddots & \\ \vdots & \iddots & & \\ G_{n-1} & & & \end{bmatrix}, \tag{15}$$

where

$$G_i = \begin{bmatrix} X_{0,i+1} & & \\ \vdots & \ddots & \\ X_{m-1,i+1} & \cdots & X_{0,i+1} \\ X_{m,i+1} & \cdots & X_{1,i+1} \\ & \ddots & \vdots \\ & & X_{m,i+1} \end{bmatrix}, \tag{16}$$

and

$$X_{p,q} = \begin{bmatrix} 0 & -c_{p,q} & b_{p,q} \\ c_{p,q} & 0 & -a_{p,q} \\ -b_{p,q} & a_{p,q} & 0 \end{bmatrix}.$$

Hence the real size of G_i is $6m \times 3m$.

4.2 From Sylvester to Cayley

There is a $6mn \times 2mn$ matrix $F_{m,n}$ such that

$$S_{m,n}^{s>t} F_{m,n} = \begin{bmatrix} C_{m,n} \\ 0 \end{bmatrix}. \tag{17}$$

The matrix $F_{m,n}$ has the following block structure

$$F_{m,n} = \begin{bmatrix} F_0 & F_1 & \cdots & F_{2m-1} \\ F_1 & \iddots & \iddots & \\ \vdots & \iddots & & \\ F_{2m-1} & & & \end{bmatrix}, \tag{18}$$

where each block F_i is a $3n \times n$ matrix. It is proved in [Chionh *et al* 1998a] that

$$F_k = (M_{k+1})^T, \qquad k = 0, \cdots, 2m-1, \tag{19}$$

where the M_k are the blocks of $M_{m,n}^{s>t}$ defined in Section 2.2. Since M_{2m} is Bezoutian, where the three Bezout matrices for $\{g_m, h_m\}$, $\{h_m, f_m\}$, $\{f_m, g_m\}$ interleave column by column, and since each Bezout matrix is symmetric [Goldman *et al* 1984], we conclude that $F_{2m-1}[= (M_{2m})^T]$ is also Bezoutian in the sense that the three Bezout matrices for $\{g_m, h_m\}$, $\{h_m, f_m\}$, $\{f_m, g_m\}$ interleave row by row.

4.3 From Mixed Cayley-Sylvester to Cayley

There is a $3mn \times 2mn$ matrix $E_{m,n}$ such that

$$M_{m,n}^{s>t} E_{m,n} = \begin{bmatrix} C_{m,n}^T \\ 0 \end{bmatrix}. \tag{20}$$

The matrix $E_{m,n}$ has the following block structure:

$$E_{m,n} = \begin{bmatrix} E_0 & E_1 & \cdots & E_{m-1} \\ E_1 & \iddots & \iddots & \\ \vdots & \iddots & & \\ E_{m-1} & & & \end{bmatrix}, \tag{21}$$

where each E_i is a martix of size $3n \times 2n$. It is proved in [Chionh *et al* 1998a] that

$$E_i = -(S_{i+1})^T, \qquad 0 \le i \le m-1, \tag{22}$$

where the S_i are the blocks of $S_{m,n}^{s>t}$ defined in Section 2.1.

5 Hybrids of the Sylvester and Cayley Resultants

Below we are going to derive the hybrid matrices of $S_{m,n}^{s>t}$ and $C_{m,n}$. To proceed, we adopt the following matrix notation: I_k is the identity matrix of order k,

$$U_k = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad V_k = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad W_k = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

That is, U_k is generated by inserting a row of zeros before every two rows in I_{2k}; V_k is generated by inserting two rows of zeros before every row of I_k; W_k is generated by inserting two rows of zeros after every row of I_k. Thus, U_k is of size $3k \times 2k$, while V_k and W_k are of size $3k \times k$.

Now recall that

$$S_{m,n}^{s>t} \cdot F_{m,n} = \begin{bmatrix} C_{m,n} \\ 0 \end{bmatrix},$$

and $F_{m,n}$ has a simple block symmetric structure [c.f. Section 4.2]. Let

$$F_j^i = \begin{bmatrix} F_i \\ \vdots \\ F_j \end{bmatrix},$$

and form the $6mn \times 6mn$ matrix

$$T_1 = \begin{bmatrix} I_{6mn-3n} & F_{2m-2}^0 & 0 \\ 0 & F_{2m-1} & U_n \end{bmatrix}_{6mn \times 6mn}.$$

Multiplying $S_{m,n}^{s>t}$ by T_1, we get

$$S_{m,n}^{s>t} \cdot T_1 = \begin{bmatrix} S_0 & & & & B_{0,0} & \\ \vdots & \ddots & & & \vdots & \\ S_{m-1} & & \ddots & & B_{m-1,0} & \\ S_m & & & \ddots & & \\ & \ddots & & S_0 & & \\ & & \ddots & \vdots & & S_0 \cdot U_n \\ & & & \ddots \; S_{m-1} & & \vdots \\ & & & S_m & & S_{m-1} \cdot U_n \\ & & & 0 & & S_m \cdot U_n \end{bmatrix}, \tag{23}$$

where the left part contains the first $2m-1$ columns (of blocks S_i) of $S_{m,n}^{s>t}$, while the middle part consists of the first column (of blocks $B_{i,j}$) of $C_{m,n}$, and the right part consists of the product of the last column of $S_{m,n}^{s>t}$ with U_n. Note, in particular, that the lower-right corner $2n \times 2n$ submatrix $S_m \cdot U_n$ is S_m with the f_m−columns dropped. Thus $S_m \cdot U_n$ is exactly the Sylvester matrix of the polynomials g_m, h_m [c.f. Section 2.1]; that is, $S_m \cdot U_n = Syl(g_m, h_m)$.

Let H_1 be the top-left corner submatrix of the right hand side of Equation (23):

$$H_1 = \begin{bmatrix} S_0 & & & & B_{0,0} \\ \vdots & \ddots & & & \vdots \\ S_{m-1} & & \ddots & & B_{m-1,0} \\ S_m & & & \ddots & \\ & \ddots & & S_0 & \\ & & \ddots & \vdots & \\ & & & \ddots \; S_{m-1} & \\ & & & S_m & \end{bmatrix}.$$

We are now going to show that $|H_1|$ is the resultant of the original polynomials $f(s,t), g(s,t), h(s,t)$. Recall that each S_i is of size $2n \times 3n$, and each $B_{i,j}$ is of size $2n \times n$. Therefore H_1 is a square matrix of order $6mn - 2n$. Thus we can rewrite Equation (23) as

$$S_{m,n}^{s>t} \cdot T_1 = \begin{bmatrix} H_1 & * \\ 0 & Syl(g_m, h_m) \end{bmatrix}.$$

It follows that

$$|S_{m,n}^{s>t}| \cdot |T_1| = |H_1| \cdot |Syl(g_m, h_m)|. \tag{24}$$

But by construction, $|T_1| = |[\, F_{2m-1} \quad U_n \,]|$. Recall from Section 4.2 that $F_{2m-1} = M_{2m}^T$ is a Bezoutiant. Now by the construction of U_n, we need only consider the rows of F_{2m-1} that do not overlap with the non-zero rows of U_n when computing $|[\, F_{2m-1} \quad U_n \,]|$. These rows of F_{2m-1} form exactly the Bezout matrix of g_m, h_m [c.f. Section 4.2]. Therefore

$$|T_1| = |[\, F_{2m-1} \quad U_n \,]| = \pm\, |Bezout(g_m, h_m)|. \tag{25}$$

Since we know that the Bezout resultant and the Sylvester resultant of two univariate polynomials are the same up to sign, we conclude from Equations (24) and (25) that $|S_{m,n}^{s>t}|$ and $|H_1|$ can only differ by at most a sign. Thus replacing the last column (of blocks S_i) of $S_{m,n}^{s>t}$ by the first column (of blocks $B_{i,j}$) of $C_{m,n}$ gives us a hybrid resultant H_1 of the Sylvester resultant and the Cayley resultant of order $6mn-2n$.

Examining the block structure of $F_{m,n}$, we see that the first column of blocks of $C_{m,n}$ depends on all the columns of $S_{m,n}^{s>t}$, but the second column of blocks of $C_{m,n}$, i.e. the coefficients of $\left[\alpha^1\beta^v, \cdots, \alpha^1\beta^{n-1}\right]$, depends only on the first $2m-1$ columns of blocks of $S_{m,n}^{s>t}$, $Ls^i\left[1, \cdots, t^{n-1}\right]$, $i = 0, \cdots, 2m-2$. Therefore we can continue the previous process, replacing the column of H_1 with index $(2m-2)$, i.e. $Ls^{2m-2}\left[1, \cdots, t^{n-1}\right]$, by the second column of $C_{m,n}$, i.e. the coefficients of $[\alpha^1\beta^o, \cdots, \alpha^1\beta^{n-1}]$. That is, there exists a transformation matrix T_2,

$$T_2 = \begin{bmatrix} I_{6mn-6n} & 0 & F_{2m-2}^1 & 0 \\ 0 & 0 & F_{2m-1} & U_n \\ 0 & I_n & 0 & 0 \end{bmatrix}, \tag{26}$$

such that

$$H_1 \cdot T_2 = \begin{bmatrix} H_2 & * \\ 0_{2n\times(6mn-4n)} & Syl(g_m, h_m) \end{bmatrix}, \tag{27}$$

where H_2 is the square matrix of order $6mn-4n$, consisting of the first $2m-2$ columns of $S_{m,n}^{s>t}$ and the first 2 columns of $C_{m,n}$. Notice that by Equation (26) we again have

$$|T_2| = |[F_{2m-1}U_n]| = \pm\, |Bezout(g_m, h_m)|.$$

Since $|H_1| \cdot |T_2| = |H_2| \cdot |Syl(g_m, h_m)|$ by Equation (27), it follows that $|H_2| = \pm\, |H_1| = \pm\, |S_{m,n}^{s>t}|$. Continuing in this fashion, we obtain $2m-1$ hybrids of the Sylvester and Cayley resultants.

After $2m$ repetitions of this process, we arrive at the Cayley matrix. This gives a direct proof that $|S_{m,n}^{s>t}| = \pm\, |C_{m,n}|$ without appealing to the properties of resultants. By the way, Dixon's fourth resultant matrix, which we mentioned in Section 1, is similar to but not the same as the $m-$th hybrid in this sequence: the $m-$th hybrid has mn columns from $C_{m,n}$ which represent polynomials of degree $m-1$ in s and $2n-1$ in t, whereas Dixon's fourth resultant has mn columns which represent polynomials of degree $2m-1$ in s and $2n-1$ in t.

6 Hybrids of the Sylvester and Mixed Cayley-Sylvester Resultants

In this section, we are going to derive hybrids of the Sylvester and the mixed Cayley-Sylvester resultants. We will replace each column (of blocks of S_i^*) of $S_{m,n}^{t>s}$ by one column (of blocks of $K_{i,j}^*$) from $M_{m,n}^{t>s}$, and decrease the order of the resulting matrix by $3m$ without changing the determinant (up to sign).

Recall that

$$S_{m,n}^{t>s} \cdot G_{m,n} = \begin{bmatrix} M_{m,n}^{t>s} \\ 0 \end{bmatrix},$$

and $G_{m,n}$ has a simple block symmetric structure [c.f. Section 4.1]. Let

$$G_j^i = \begin{bmatrix} G_i \\ \vdots \\ G_j \end{bmatrix},$$

and form the $6mn \times 6mn$ matrix

$$T_1 = \begin{bmatrix} I_{6mn-6m} & G_{n-2}^0 & 0 \\ 0 & G_{n-1} & \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \end{bmatrix}_{6mn\times 6mn}.$$

Multiplying $S_{m,n}^{t>s}$ by T_1, we get

$$S_{m,n}^{t>s} \cdot T_1 = \begin{bmatrix} S_0^* & & & K_{0,0}^* & \\ \vdots & \ddots & & \cdots & \\ S_{n-2}^* & \cdots & S_0^* & K_{n-2,0}^* & \\ S_{n-1}^* & \cdots & S_1^* & K_{n-1,0}^* & S_0^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \\ S_n^* & \cdots & S_2^* & & S_1^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \\ & \ddots & \vdots & & \vdots \\ & & S_n^* & & S_{n-1}^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \\ & & 0 & & S_n^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \end{bmatrix}, \tag{28}$$

where the left part consists of the first $n-1$ columns (of blocks S_i^*) of $S_{m,n}^{t>s}$, while the middle part contains the first column of blocks of $M_{m,n}^{t>s}$, and the right part consists of the product of the last column of blocks of $S_{m,n}^{t>s}$ with $\begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix}$. Let H_1 be the top-left corner submatrix of the right hand side of Equation (28):

$$H_1 = \begin{bmatrix} S_0^* & & & K_{0,0}^* \\ \vdots & \ddots & & \vdots \\ S_{n-2}^* & \cdots & S_0^* & K_{n-2,0}^* \\ S_{n-1}^* & \cdots & S_1^* & K_{n-1,0}^* \\ S_n^* & \cdots & S_2^* & \\ & \ddots & \vdots & \\ & & S_n^* & \end{bmatrix}.$$

Since each S_i^* is of size $3m \times 6m$, and each $K_{i,j}^*$ is of size $3m \times 3m$, H_1 is a square matrix of order $6mn - 3m$. Now we can rewrite Equation (28) as

$$S_{m,n}^{t>s} \cdot T_1 = \begin{bmatrix} H_1 & * \\ 0 & S_n^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \end{bmatrix}.$$

It follows that

$$|S_{m,n}^{s>t}| \cdot |T_1| = |H_1| \cdot \left| S_n^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \right|. \tag{29}$$

But recall that S_n^* consists of the coefficients of the monomials $[s^v f_n^*, s^v g_n^*, s^v h_n^*]$, $v = 0, \cdots, 2m-1$; that is,

$$S_n^* = \begin{bmatrix} L_{0,n} & & & & \\ \vdots & \ddots & & & \\ L_{m-1,n} & \cdots & L_{0,n} & & \\ L_{m,n} & \cdots & L_{1,n} & L_{0,n} & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & L_{m,n} & L_{m-1,n} & \cdots & L_{0,n} \\ & & & L_{m,n} & \cdots & L_{1,n} \\ & & & & \ddots & \vdots \\ & & & & & L_{m,n} \end{bmatrix},$$

where $L_{i,j} = [a_{i,j} \;\; b_{i,j} \;\; c_{i,j}]$. Therefore,

$$S_n^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} = \begin{bmatrix} c_{0,n} & & & & & & & & \\ \vdots & \ddots & & & & & & & \\ c_{m-1,n} & \cdots & c_{0,n} & & & & & & \\ c_{m,n} & \cdots & c_{1,n} & b_{0,n} & c_{0,n} & & & & \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & & \\ & & c_{m,n} & b_{m-1,n} & c_{m-1,n} & \cdots & \cdots & b_{0,n} & c_{0,n} \\ & & & b_{m,n} & c_{m,n} & \cdots & \cdots & b_{1,n} & c_{1,n} \\ & & & & & \ddots & \ddots & \vdots & \vdots \\ & & & & & & & b_{m,n} & c_{m,n} \end{bmatrix},$$

so it is easy to see that

$$\left| S_n^* \cdot \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \right| = (c_{0,n})^m \cdot |Syl(g_n^*, h_n^*)|.$$

Hence by Equation (29), we have

$$|S_{m,n}^{t>s}| \cdot |T_1| = \pm\, |H_1| \cdot (c_{0,n})^m \cdot |Syl(g_n^*, h_n^*)|. \tag{30}$$

Now notice that $|T_1| = \left| \begin{bmatrix} G_{n-1} & \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \end{bmatrix} \right|$. Moreover, by the construction of V_m and U_m, we need only consider the rows of G_{n-1} that do not overlap with the nonzero rows of V_m and U_m when we compute the determinant of $\begin{bmatrix} G_{n-1} & \begin{bmatrix} V_m & 0 \\ 0 & U_m \end{bmatrix} \end{bmatrix}$. That is, we only care about the rows $3j$, $3j+1$, $j = 0, \cdots, m-1$, and the rows $3m + 3k$, $k = 0, \cdots, m-1$, of G_{n-1}. Collecting these $3m$ rows of G_{n-1}, we get the following $3m \times 3m$ matrix [c.f. Equation (16)]:

$$\begin{bmatrix} 0 & -c_{0,n} & b_{0,n} & 0 & 0 & 0 & \cdots & 0 & 0 \\ c_{0,n} & 0 & -a_{0,n} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -c_{1,n} & b_{1,n} & 0 & -c_{0,n} & b_{0,n} & \cdots & 0 & 0 \\ c_{1,n} & 0 & -a_{1,n} & c_{0,n} & 0 & -a_{0,n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -c_{m,n} & b_{m,n} & 0 & -c_{m-1,n} & b_{m-1,n} & \cdots & -c_{1,n} & b_{1,n} \\ 0 & 0 & 0 & 0 & -c_{m,n} & b_{m,n} & \cdots & -c_{2,n} & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -c_{m,n} & b_{m,n} \end{bmatrix}.$$

Rearrange the rows and columns of this matrix as follows: group the columns indexed by $3j$, $j = 0, \cdots, m-1$, together and collect the rows indexed by $2j+1$, $j = 0, \cdots, m-1$, together, and put them at the top-left corner. Then we get the matrix

$$\begin{bmatrix} c_{0,n} & 0 & \cdots & 0 & -a_{0,n} & 0 & 0 & 0 & \cdots & 0 & 0 \\ c_{1,n} & c_{0,n} & \cdots & 0 & 0 & -a_{1,n} & 0 & -a_{0,n} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ c_{m-1,n} & c_{m-2,n} & \cdots & c_{0,n} & 0 & -a_{m-1,n} & 0 & -a_{m-2,n} & \cdots & 0 & -a_{0,n} \\ 0 & 0 & \cdots & 0 & -c_{0,n} & b_{0,n} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -c_{1,n} & b_{1,n} & -c_{0,n} & b_{0,n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -c_{m-1,n} & b_{m-1,n} & -c_{m-2,n} & b_{m-2,n} & \cdots & -c_{0,n} & b_{0,n} \\ 0 & 0 & \cdots & 0 & -c_{m,n} & b_{m,n} & -c_{m-1,n} & b_{m-1,n} & \cdots & -c_{1,n} & b_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & -c_{m,n} & b_{m,n} \end{bmatrix}.$$

It is easy to see that the determinant of this matrix is equal to $\pm\,(c_{0,n})^m \cdot |Syl(g_n^*, h_n^*)|$. Therefore by Equation (30)

$$|S_{m,n}^{t>s}| \cdot (c_{0,n})^m \cdot |Syl(g_n^*, h_n^*)| = \pm\,|H_1| \cdot (c_{0,n})^m \cdot |Syl(g_n^*, h_n^*)|.$$

Hence $|S_{m,n}^{t>s}|$ and $|H_1|$ are the same up to sign. Thus, we get a hybrid resultant matrix H_1 of the Sylvester and Cayley-Sylvester resultants with order $6mn - 3m$.

Checking the block structure of $G_{m,n}$, we see that the column of blocks of $M_{m,n}^{t>s}$ with index $(n-1-j)$ depends on the columns of blocks of $S_{m,n}^{t>s}$ with indices $[0, \cdots, j]$. Therefore, we can continue this process, replacing the column of blocks of $S_{m,n}^{t>s}$ indexed by $(n-1-j)$, i.e., $Lt^{n-1-j}\,[\,1 \quad \cdots \quad s^{2m-1}\,]$, by the column of blocks of $M_{m,n}^{t>s}$ indexed by j, i.e. $\overline{L}_j\,[\,1 \quad \cdots \quad s^{m-1}\,]$, for $j = 0, \cdots, n-1$. For example, repeating the previous process, i.e., replacing the $(n-1)$st column of blocks of H_1, i.e. $Lt^{n-2}\,[\,1 \quad \cdots \quad s^{2m-1}\,]$ by the second column of blocks of $M_{m,n}^{t>s}$, i.e. $\overline{L}_1\,[\,1 \quad \cdots \quad s^{m-1}\,]$, we get a second hybrid resultant matrix H_2 of the Sylvester and Cayley-Sylvester resultants with order $6mn - 6m$.

After n repetitions of this process, we arrive at the mixed Cayley-Sylvester resultant. Again this provides a direct proof that $|S_{m,n}^{t>s}| = \pm\,|M_{m,n}^{t>s}|$ without appealing to the properties of resultants.

7 Hybrids of the Mixed Cayley-Sylvester and Cayley Resultants

In this section we present the construction of hybrid resultants that consist of some columns of the mixed Cayley-Sylvester matrix and some columns of the transposed Cayley matrix. The derivation in this section is very similar to those of Section 5 and Section 6.

Recall that

$$M_{m,n}^{s>t} \cdot E_{m,n} = \begin{bmatrix} C_{m,n}^T \\ 0 \end{bmatrix},$$

and $E_{m,n}$ has a simple block symmetric structure [c.f. Section 4.3]. Let

$$E_j^i = \begin{bmatrix} E_i \\ \vdots \\ E_j \end{bmatrix},$$

and write

$$M_{m,n}^{s>t} = [\; K_0 \quad K_1 \quad \cdots \quad K_{m-1} \;],$$

where each K_j, $0 \le j \le m-1$, consists of a column of blocks M_i. Let

$$K_j^i = [\; K_i \quad \cdots \quad K_j \;].$$

Since $E_{m,n}$ is block symmetric and upper triangular,

$$\begin{aligned} M_{m,n}^{s>t} E_{m,n} &= [\; K_{m-1}^0 E_{m-1}^0 \quad K_{m-2}^0 E_{m-1}^1 \quad \cdots \quad K_0^0 E_{m-1}^{m-1} \;] \\ &= \left[\begin{matrix} \begin{bmatrix} B_{0,0}^T \\ \vdots \\ B_{0,2m-1}^T \end{bmatrix} & \begin{bmatrix} B_{1,0}^T \\ \vdots \\ B_{1,2m-1}^T \end{bmatrix} & \cdots & \begin{bmatrix} B_{m-1,0}^T \\ \vdots \\ B_{m-1,2m-1}^T \end{bmatrix} \\ 0 & 0 & \cdots & 0 \end{matrix}\right]. \end{aligned}$$

Note that the bottom $(m-1-j) \cdot n$ rows of K_j^0 are all zero. Moreover, $K_{m-1-j}^0 E_{m-1}^j$, $j = 0, \cdots, m-1$, has at most $2mn$ non-zero rows, since

$$K_{m-1-j}^0 E_{m-1}^j = \begin{bmatrix} B_{j,0}^T \\ \vdots \\ B_{j,2m-1}^T \\ 0 \end{bmatrix}.$$

The j−th hybrid matrix H_j is given by

$$\begin{aligned}\begin{bmatrix} H_j \\ 0 \end{bmatrix} &= \begin{bmatrix} K^0_{m-1-j} & K^0_{m-1}E^0_{m-1} & K^0_{m-2}E^1_{m-1} & \cdots & K^0_{m-j}E^{j-1}_{m-1} \end{bmatrix} \\ &= \begin{bmatrix} K^0_{m-1-j} & \begin{bmatrix} B^T_{0,0} \\ \vdots \\ B^T_{0,2m-1} \end{bmatrix} & \begin{bmatrix} B^T_{1,0} \\ \vdots \\ B^T_{1,2m-1} \end{bmatrix} & \cdots & \begin{bmatrix} B^T_{j-1,0} \\ \vdots \\ B^T_{j-1,2m-1} \end{bmatrix} \\ & 0 & 0 & & 0 \end{bmatrix}\end{aligned} \tag{31}$$

for $0 \le j \le m$. That is, H_j consists of the first $m-j$ columns of blocks of $M^{s>t}_{m,n}$ and the first j columns of blocks of $C^T_{m,n}$, with the bottom $n \cdot j$ rows of zeros truncated. Note that the matrix H_j has $(3m-j)\cdot n$ rows and $3n(m-j) + 2n \cdot j = n(3m-j)$ columns. Hence H_j is a square matrix. In particular, H_0 is $M^{s>t}_{m,n}$, and H_m is $C^T_{m,n}$.

To show that $|H_j|$ and $|H_{j+1}|$ are equal up to sign, $0 \le j \le m-1$, we construct the matrix

$$T_j = \begin{bmatrix} I_{3(m-j-1)n} & 0 & E^j_{m-2} & 0 \\ 0 & 0 & E_{m-1} & W_n \\ 0 & I_{2jn} & 0 & 0 \end{bmatrix},$$

where W_n is the $3n \times n$ matrix defined at the beginning of Section 5. It is easy to see that

$$|T_j| = \pm\, |\,[\, E_{m-1} \quad W_n \,]\,|.$$

By the construction of W_n, we need only consider the rows of E_{m-1} that do not overlap with the non-zero rows of W_n to compute $|\,[\, E_{m-1} \quad W_n \,]\,|$. Since $E_{m-1} = -S^T_m$ [c.f. Equation (22)], and since S_m is Sylvester-like (Section 2.1), $|\,[\, E_{m-1} \quad W_n \,]\,|$ is exactly the Sylvester resultant of the univariate polynomials g_m, h_m defined in Section 2.1. Therefore,

$$|T_j| = \pm\, Syl(g_m, h_m). \tag{32}$$

Moreover,

$$\begin{aligned}\begin{bmatrix} H_j \\ 0 \end{bmatrix} \cdot T_j &= \begin{bmatrix} K^0_{m-1-j} & K^0_{m-1}E^0_{m-1} & K^0_{m-2}E^1_{m-1} & \cdots & K^0_{m-j}E^{j-1}_{m-1} \end{bmatrix} \cdot T_j \\ &= \begin{bmatrix} K^0_{m-1-j}\begin{bmatrix} I \\ 0 \end{bmatrix} & K^0_{m-1}E^0_{m-1} & \cdots & K^0_{m-j}E^{j-1}_{m-1} & K^0_{m-j-1}E^j_{m-1} & K^0_{m-j-1}\begin{bmatrix} 0 \\ W_n \end{bmatrix} \end{bmatrix}\end{aligned} \tag{33}$$

But

$$\begin{aligned} K^0_{m-1-j}\begin{bmatrix} I_{3(m-1-j)n} \\ 0 \end{bmatrix} &= [\, K^0_{m-2-j} \quad K_{m-1-j} \,] \cdot \begin{bmatrix} I_{3(m-1-j)n} \\ 0 \end{bmatrix} \\ &= K^0_{m-2-j}, \end{aligned}$$

and

$$\begin{aligned} K^0_{m-1-j}\begin{bmatrix} 0 \\ W_n \end{bmatrix} &= [\, K^0_{m-2-j} \quad K_{m-1-j} \,] \cdot \begin{bmatrix} 0 \\ W_n \end{bmatrix} \\ &= K_{m-1-j} \cdot W_n. \end{aligned}$$

Hence,

$$\begin{bmatrix} H_j \\ 0 \end{bmatrix} \cdot T_j = [\, K^0_{m-2-j} \quad K^0_{m-1}E^0_{m-1} \quad \cdots \quad K^0_{m-j}E^{j-1}_{m-1} \quad K^0_{m-j-1}E^j_{m-1} \quad K_{m-j-1}W_n \,]. \tag{34}$$

Now let us focus on the top $n \cdot (3m - j)$ rows of the right hand side of Equation (34), since all the rows below are zero. Notice that the top-left corner $[n \cdot (3m - j - 1)] \times [n \cdot (3m - j - 1)]$ submatrix is exactly H_{j+1}. Furthermore, the last n non-zero rows contain all zeros except for $M_{2m} \cdot W_n$, which arises from the last n columns of $K_{m-j-1} \cdot W_n$. Since M_{2m} is a Bezoutiant [c.f. Section 2.2], by the construction of W_n, the product $M_{2m} \cdot W_n$ is the Bezout matrix of g_m, h_m. Therefore, Equation (34) implies that

$$H_j \cdot T_j = \begin{bmatrix} H_{j+1} & * \\ 0 & Bezout(g_m, h_m) \end{bmatrix}.$$

Hence

$$|H_j| \cdot |T_j| = |H_{j+1}| \cdot |Bezout(g_m, h_m)|. \tag{35}$$

By Equation (32) and Equation (35) we conclude that

$$|H_j| = \pm\, |H_{j+1}|, \qquad j = 0, \cdots, m-1.$$

Thus we generate a sequence of $m-1$ hybrid resultants of the Cayley and the mixed Cayley-Sylvester resultants. Since $H_0 = M^{s>t}_{m,n}$ and $H_m = C^T_{m,n}$, we obtain in this fashion a direct proof that $|M^{s>t}_{m,n}| = \pm\, |C_{m,n}|$ without appealing to the properties of resultants.

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