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Abstract

Equational rewriting is an important topic in the field of term rewriting systems. This extends ordinary rewriting, and is important especially in cases where some of the equations cannot be oriented into a terminating set of rewrite rules. The most widely used form of equational rewriting is associative-commutative (AC) rewriting, to handle operators like $+$ (addition) and $\wedge$ (and).

There are several ways to extend ordinary rewriting to handle "built-in" equational theories. We discuss two of these, namely R/E and $E \setminus R$. We have found that fine nuances between these different rewrite relations are often hard to understand. The same is often the case with important concepts like *coherence.*

To clarify some of these issues, we put together counterexamples to many possible conjectures in this field. It is hoped that this collection of examples will be useful to researchers in equational reasoning.

1 Introduction

E-rewriting and E-unification have gained a lot of momentum these days in the field of term rewriting systems. In their foundational paper on term rewriting systems Knuth and Bendix observed that certain equational axioms like commutativity, $x + y = y + x$, cannot be oriented into a rewrite rule. They emphasised the need for extending their procedure to handle such axioms [9].

This was taken up by Lankford and Ballantyne in the context of permutative theories [7] and, in particular, of the theory of associativity and commutativity (AC) [8]. Later Peterson and Stickel gave an extension of the Knuth-Bendix procedure to handle arbitrary equational theories with certain restrictions, and applied it to theories with AC symbols [12]. (This is now often referred to as the ACKB procedure.) But a clear foundation of the concept of E-rewriting (for general E) and its variants was only given by Jouannaud [3] and by Kirchner in her Ph. D. thesis [6] and in her paper with Jouannaud [4]. Their work carefully distinguished between the relations R/E and $E \backslash R$, and defined the elegant general concept of E-coherence (defined in the next section) generalizing the notion of E-compatibility defined by Peterson and Stickel. The E-completion procedure in [4] was further generalized by Bachmair and Dershowitz [1]. Recently Marché has developed a new approach to E-rewriting to handle situations where R/E is not noetherian [10, 11]. (See [2] for a survey.)

Over the years we (and several others) have found the concepts very hard to internalize. One of the problems is that only the case where E = AC has been investigated in detail. There have some recent attempts with $E = AC1$ [5]. Experience with other equational theories has been lacking. As a result, many misleading and erroneous statements have crept into even some excellent papers.

In this report, we put together counterexamples to many possible conjectures. One of these (Lemma 5.3), showing that termination of $E\backslash R$ is not enough to ensure the Church-Rosser property, was anticipated by Jouannaud and Kirchner [4], but no concrete counterexample was given there. Some others were found in papers on E-rewriting and some are our own conjectures.

We hope this collection of examples will be useful to researchers of equational rewriting.

2 Preliminaries

We briefly explain the basic notions of rewrite systems. An interested reader may consult [2] for a survey of the area.

2.1 Rewrite Systems

Let $T(F, X)$ denote the set of terms constructed from a (finite) set F of function symbols and a (countable) set X of variables. We normally use the letters a through h for function symbols; s, t, and u through w for arbitrary terms; x, y, and z for variables. Each function symbol $f \in F$ has an *arity* $n \geq 0$ which is the number of arguments (immediate subterms) that it has in a well-formed term. *Constants* are function symbols of arity zero. Variable-free terms are called *ground*.

A term t in $T(F, X)$ may be viewed as a finite ordered tree. Internal nodes are labeled with function symbols (from F) of arity greater than 0. The outdegree of an internal node is the same as the arity of the label. Leaves are labeled with either variables (from X) or constants. The size of a term is the number of nodes in the tree. We use $u[t]_\pi$ to denote a term that has t as a subterm at position π. We use $u[\cdot]$ to denote the *context* in which t occurs in the term $u[t]_\pi$. The context is the tree obtained by deleting t from the tree. By $t|_\pi$, we denote the *subterm* of t rooted at *position* π. A subterm of t is called *proper* if it is distinct from t.

A substitution σ is a mapping from variables to terms such that $x\sigma \neq x$ for a finite number of variables x's. A substitution can be extended to be a mapping from terms (equations) to terms (equations).

A rewrite *rule* over a set T of terms is an ordered pair (l, r) of terms such that the variables in r also appear in l, and is written $l \to r$. A *rewrite system* (or *term rewriting system*) R is a finite set of such rules.

We use $\to$ (sometimes $\to_R$) to denote the rewrite relation associated with a rewriting system R. A term s *rewrites* to another term t, denoted $s \to t$, if $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$, for some rule $l \to r$ in R, position π in s, and substitution σ. The rewrite relation $\to$ over T is *terminating* (or *well-founded*) if there is no infinite sequence of terms t_i in T such that $t \to t_1 \to t_2 \cdots$.

$\to_R^*$ denotes the reflexive transitive closure of the rewrite relation $\to_R$. We use $\leftrightarrow_R^*$ to denote its reflexive symmetric transitive closure. $\circ$ denotes the composition of two relations.

A rewrite relation $\to_R$ is said to be locally confluent iff $\leftarrow_R \circ \to_R \subseteq \to_R^* \circ \leftarrow_R^*$. A rewrite system is said to be confluent iff $\leftarrow_R^* \circ \rightarrow_R^* \subseteq \rightarrow_R^* \circ \leftarrow_R^*$ and convergent if it is terminating and confluent. A rewrite relation has Church-Rosser property iff $\leftrightarrow_R^* = \rightarrow_R^* \circ \leftarrow_R^*$. We abbreviate this by stating that $\rightarrow_R$ is Church-Rosser.

The rewrite relation $\rightarrow_R$ can be extended to rewrite over congruence classes defined by a set of equations E, rather than terms. These constitute E-rewriting systems. An E-rewrite system consists of a set of equations E and a set of rules R.

The rewrite relation $\rightarrow_{R/E}$ over the congruence classes of E is defined as $s \rightarrow_{R/E} t$ iff

$$s \leftrightarrow_E^* s', \quad s'|_p = l\sigma, \quad s'[r\sigma]_p \leftrightarrow_E^* \quad t$$

for a substitution σ, position p in s' and rule $l \rightarrow_R r$ in R.

A restricted version of $\rightarrow_{R/E}$, denoted by $E \setminus R$, which uses only E-matching unlike the former[1], for the application of the rules is defined as, $s \rightarrow_{E \setminus R} t$ iff there exists a position p in s such that

$$s|_p \leftrightarrow_E^* l\sigma, \;\; t = s[r\sigma]_p$$

The properties associated with the rewrite relation $\rightarrow_R$ such as local confluence, can be extended naturally to equational rewriting. Let $\rightarrow_T$ be any rewrite relation T such that $\rightarrow_R \subseteq \rightarrow_T \subseteq \rightarrow_{R/E}$.

The rewrite relation $\rightarrow_T$ is said to be locally confluent modulo E with respect to R iff $\leftarrow_R \circ \rightarrow_T \subseteq \rightarrow_T^* \circ \leftrightarrow_E^* \circ \leftarrow_T^*$. The rewrite relation $\rightarrow_T$ is confluent modulo E iff $\leftarrow_R^* \circ \rightarrow_T^* \subseteq \rightarrow_T^* \circ \leftrightarrow_E^* \circ \leftarrow_T^*$.

The rewrite relation $\rightarrow_T$ is said to be locally coherent modulo E iff $\leftrightarrow_E \circ \rightarrow_T \subseteq \rightarrow_T^* \circ \leftrightarrow_E^* \circ \leftarrow_T^*$. The rewrite relation $\rightarrow_T$ is coherent modulo E iff $\leftrightarrow_E^* \circ \rightarrow_T^* \subseteq \rightarrow_T^* \circ \leftrightarrow_E^* \circ \leftarrow_T^*$.

The rewrite relation $\rightarrow_T$ is said to be Church-Rosser modulo E iff $\leftrightarrow_{E \cup R}^* \subseteq \rightarrow_T^* \circ \leftrightarrow_E^* \circ \leftarrow_T^*$.

(The phrase "modulo E" is often omitted when it is obvious from the context. Note that we are particularly interested in $\rightarrow_T$ being $\rightarrow_{E \setminus R}$ in the rest of the sections of this report.)

3 Critical Pair Criteria

The following critical pair criteria were given by Jouannaud and Kirchner [4] for checking whether $\rightarrow_{E \setminus R}$ is Church-Rosser modulo E. By $CSU_E(s,t)$ we mean a complete set of equational unifiers (also called "semantic unifiers") of s and t modulo E.

A complete set of E-critical pairs of a rule $l \rightarrow r$ *on* a rule $g \rightarrow d$ is defined as

$$CP_E(l \rightarrow r, g \rightarrow d) = \{< \theta(g[r]_p), \theta(d) >| \; \theta \in CSU_E(g|_p, l)\}$$

where p is a non-variable position in g. Now

$$CP_E(l \rightarrow r, g = d) = CP_E(l \rightarrow r, g \rightarrow d) \cup CP_E(l \rightarrow r, d \rightarrow g)$$

and similarly

[1] Dershowitz and Jouannaud [2] have suggested the terms "equational-rewriting" for R/E and "extended rewriting" for $E \setminus R$.

$$CP_E(l = r, g \to d) = CP_E(l \to r, g \to d) \cup CP_E(r \to l, g \to d)$$

We use $CP_E(R, R)$ to denote the set of E-critical pairs with all $l \to r \in R$ and all $g \to d \in R$; $CP_E(R, E)$ denotes the set of critical pairs obtained from all $l \to r \in R$ and all $g = d \in E$; the denotation of $CP_E(E, R)$ follows by symmetry.

(The subscript E is omitted when there is no equational theory.)

A critical pair $< s,\ t >$ is trivial if there exist terms s' and t' such that $s \ \to^*_{E\backslash R} \ s'$, $t \ \to^*_{E\backslash R} \ t'$ and $s' \ \leftrightarrow^*_E \ t'$.

Lemma 3.1 *If R/E is terminating, then $\to_{E\backslash R}$ is Church-Rosser modulo E if and only if all critical pairs in $CP_E(R, R)$ and $CP_E(R, E)$ are trivial.*

4 Properties

Lemma 4.1 *If $R_1 \cup R_2$ is Church-Rosser then R_1/E_2 is Church-Rosser modulo E_2, where E_2 is obtained from R_2 by treating the rules as equations.*

Proof.

$$\begin{array}{rcl} \to^*_{R_1 \cup R_2} & = & (\to^*_{R_2} \circ \to_{R_1} \circ \to^*_{R_2})^* \cup \to^*_{R_2} \\ & \subseteq & (\leftrightarrow^*_{E_2} \circ \to_{R_1} \circ \leftrightarrow^*_{E_2})^* \cup \leftrightarrow^*_{E_2} \\ & = & \to^*_{R_1/E_2} \cup \leftrightarrow^*_{E_2} \end{array}$$

Thus

$$\to^*_{R_1 \cup R_2} \circ \leftarrow^*_{R_1 \cup R_2} \quad \subseteq \quad (\to^*_{R_1/E_2} \cup \leftrightarrow^*_{E_2}) \circ (\leftrightarrow^*_{E_2} \cup \leftarrow^*_{R_1/E_2})$$

5 Counter-examples

The first two lemmas are straightforward.

Lemma 5.1 *It is not the case that if $E \setminus R$ is terminating and Church-Rosser modulo E then R/E is terminating.*

Proof. Let E be :

$$a + b = a + c$$

and R be :

$$c \to b$$

where a, b and c are constants.

R/E is not terminating as is illustrated by the following loop.

$$a + b \leftrightarrow_E a + c \rightarrow_R a + b$$

We note by the definition of $E \setminus R$ that in the present case for any term t there exists no position p in t such that $t|_p = c\sigma$, for any substitution σ, unless $t|_p = c$. Hence $\rightarrow_{E\setminus R}$ is equivalent to $\rightarrow_R$ and is terminating.

$\rightarrow_{E\setminus R}$ is clearly Church-Rosser since $\leftrightarrow_R = \leftrightarrow_{E\cup R}$ and $\rightarrow_R$ can be easily seen to be convergent.
□

The above lemma also shows that the statement of the critical pair lemma (Lemma 3.1) cannot be made an if-and-only-if statement.

Lemma 5.2 *It is not the case that if R/E is Church-Rosser then $E \setminus R$ is Church-Rosser.*

Proof. The proof proceeds by contradiction by using the following counterexample.

$$E: \quad a + (b + c) = (a + b) + c$$

$$R: \quad (a + b) \rightarrow c$$

For $E \setminus R$ to be Church-Rosser it must be the case that for any two terms s and r, if $(s, r) \in \leftrightarrow^*_{R\cup E}$ then $(s, r) \in \rightarrow^*_{E\setminus R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$. Let s be $a + (b + b)$ and r be $c + b$. $(a + (b + b), c + b) \in \leftrightarrow^*_{R\cup E}$, since $(a + (b + b)) \leftrightarrow^*_E ((a + b) + b) \rightarrow_R (c + b)$. But $(a + (b + b), c + b) \notin \rightarrow^*_{E\setminus R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$, since $(a + (b + b))$ and $(c + b)$ are irreducible with respect to $\rightarrow^*_{E\setminus R}$ and are not equivalent modulo E. Therefore $E \setminus R$ is not Church-Rosser.

The system

$$\begin{aligned} R': \quad & a + (b + c) \rightarrow (a + b) + c \\ & (a + b) \rightarrow c \end{aligned}$$

obtained by orienting the equation in E is convergent. Hence R/E is Church-Rosser modulo E by Lemma 4.1. □

Lemma 5.3 *It is not the case that if $E \setminus R$ is locally confluent and locally coherent and $E \setminus R$ is terminating, then $E \setminus R$ is Church-Rosser.*

Proof. Let E be the following set of equations over ground terms :

$$(a + b) + d = (a + c) + d$$

$$(a + c) + d = g$$

Let R be the following ground rewrite system :

$$c \rightarrow b$$

$$(a + b) \rightarrow g$$

We show subsequently that $E \setminus R$ is terminating and $E \setminus R$ is locally coherent and locally confluent and that $E \setminus R$ is not Church-Rosser.

First of all, note that $\rightarrow_{E\setminus R}$ is equivalent to $\rightarrow_R$ since for any term t, $t \leftrightarrow^*_E l$ where l is the left-hand side of a rule in R if and only if $t = l$.

- *$E \setminus R$ is terminating*: Trivial, since $\rightarrow_R$ is easily shown to be noetherian. Note that R/E is clearly not terminating due to the presence of a loop illustrated as follows

$$(a+b)+d \leftrightarrow_E (a+c)+d \rightarrow_R (a+b)+d$$

- *$E \setminus R$ is locally confluent with respect to R* :

  Since $\rightarrow_{E\setminus R}$ is equivalent to $\rightarrow_R$ it suffices to show that $\leftarrow_R \circ \rightarrow_R \subseteq \rightarrow^*_{E\setminus R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$. Since R is a reduced ground rewriting system with no critical pairs $\rightarrow_R$ is locally confluent. Hence for any two terms s and t, if $(s,t) \in \leftarrow_R \circ \rightarrow_R$ then $(s,t) \in \rightarrow^*_R \circ \leftarrow^*_R$. Since $\rightarrow^*_R \circ \leftarrow^*_R \subseteq \rightarrow^*_{E\setminus R} \circ \leftarrow^*_{E\setminus R}$, it follows that $E \setminus R$ is locally confluent.

- *$E \setminus R$ is locally coherent* :

  We have to show that $\leftarrow_{E\setminus R} \circ \leftrightarrow_E \subseteq \rightarrow^*_{E\setminus R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$. As stated earlier, $\rightarrow_{E\setminus R}$ is equivalent to $\rightarrow_R$. Hence let $(s,s') \in \leftarrow_R \circ \leftrightarrow_E$. Therefore there exists a term m such that $m \rightarrow_R s$ and $m \leftrightarrow_E s'$. Let p and q be any two positions in m such that $s = m[r]_p$ for some rule $l \rightarrow r \in R$ and $s' = m[e_2]_q$, where $e_1 = e_2 \in E$.

  If p and q are disjoint positions then it trivially follows that $(s,s') \in \rightarrow^*_{E\setminus R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$, since we can rewrite s' at position p and replace in s the subterm at position q by q'.

  If positions p and q are not disjoint, then there are two possibilities—$p \leq q$ or $q \leq p$. It is easy to see from E and R that $p \leq q$ cannot be the case since E has no effect on the left hand sides of the rules in R.

  Hence it must be that $q \leq p$. We can assume, without loss of generality, that q is the top position (i.e., $q = \lambda$) in m. It can now be shown by straightforward case analysis that in such a situation $(s,s') \in \rightarrow^*_R \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$.

  1. $m|_\lambda = m = (a+b)+d$ and $s' = (a+c)+d$. Since $q \leq p$, the only redex available for rewrite is $(a+b)$. Hence $s = g+d$, by the rule $(a+b) \rightarrow_R g$. Since $s' \rightarrow_R (a+b)+d$ by the rule $c \rightarrow_R b$, which by subsequent application of the other rule reduces to $g+d$. Therefore $(s,s') \in \rightarrow^*_R \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$.
  2. $m = (a+c)+d$ and $s' = g$. Again by a similar argument as in the previous case $s = (a+b)+d$. Since $s \leftrightarrow^*_E s'$, it follows that $(s,s') \in \rightarrow^*_R \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$.
  3. $m = (a+c)+d$ and $s' = (a+b)+d$. This case is trivial since $s = (a+b)+d$.

  These are the only two possible cases as the set of equations and the rewrite rules are over ground terms. Therefore $\rightarrow_{E\setminus R}$ is locally coherent.

- *$E \setminus R$ is not Church-Rosser* : For $E \setminus R$ to be Church-Rosser it must be the case that for any two terms s and t, if $(s,t) \in \leftrightarrow^*_{R\cup E}$ then $(s,t) \in \rightarrow^*_{E\setminus R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\setminus R}$. Let s be $(g+d)$ and t be g. $(g+d, g) \in \leftrightarrow^*_{R\cup E}$, since

$$(a+c)+d \leftrightarrow_E (a+b)+d \rightarrow_R g+d \quad \textit{and} \quad (a+c)+d \leftrightarrow_E g$$

But $g+d$ and g both are irreducible with respect to $\rightarrow_{E\backslash R}$. Hence $(g+d, g) \notin \in \rightarrow^*_{E\backslash R} \circ \leftrightarrow^*_E \circ \leftarrow^*_{E\backslash R}$. Therefore $\rightarrow_{E\backslash R}$ is not Church-Rosser.

□

In contrast to Lemma 4.1,

Lemma 5.4 *It is not the case that if $R_1 \cup R_2$ is convergent then $E_2 \setminus R_1$ is Church-Rosser where E_2 is obtained from R_2 by treating the rules as equations.*

Proof. Consider the following example :

$$R_1 : g(f(x)) \rightarrow x$$

$$R_2 : g(h(x)) \rightarrow h(g(x))$$

Thus $E_2 = \{g(h(x)) = h(g(x))\}$.

We first show that $E_2 \setminus R_1$ is not Church-Rosser. For this we have to show that there exist terms s and t such that $(s,t) \in \leftrightarrow^*_{R_2 \cup R_1}$ whereas $(s,t) \notin \rightarrow^*_{R_2\backslash R_1} \circ \leftrightarrow^*_{E_2} \circ \leftarrow^*_{R_2\backslash R_1}$.

Let s be $g(h(f(x)))$ and t be $h(x)$. $(s,t) \in \leftrightarrow^*_{E_2 \cup R_1}$, since $h(g(f(x))) \leftrightarrow_{R_2} g(h(f(x)))$ and $h(g(f(x))) \rightarrow_{R_1} h(x)$. But $g(h(f(x)))$ and $h(x)$ are irreducible with respect $\rightarrow^*_{E_2\backslash R_1}$ and $g(h(f(x)))$ and $h(x)$ are not E_2 equivalent. Therefore, $(s,t) \notin \rightarrow^*_{E_2\backslash R_1} \circ \leftrightarrow^*_{E_2} \circ \leftarrow^*_{E_2\backslash R_1}$.

With the precedence on the function symbols being given as $g \succ h$, it can be easily seen using recursive path ordering that for each of the rules $l \rightarrow r$ in $R_1 \cup R_2$ that $l \succ_{rpo} r$. Hence $R_1 \cup R_2$ is terminating. Since $R_1 \cup R_2$ is terminating it suffices to show that it is locally confluent to establish that it is convergent. This is obvious since $R_1 \cup R_2$ is a reduced rewrite system with no critical pairs amongst the rules. □

Lemma 5.5 *It is not the case that if R/E is Church-Rosser then R is locally coherent with respect to E.*

Proof. The proof is based on the following counter-example[2] given below.

$$E : a = b$$

$$R : f(x,x) \rightarrow x$$

R is not locally coherent with respect to E :

Consider the term $f(b,b)$. $f(b,b) \rightarrow_R b$ and $f(b,b) \leftrightarrow_E f(a,b)$. Hence $(b, f(a,b)) \in \leftarrow_R \circ \leftrightarrow_E$. Since b and $f(a,b)$ are both irreducible with respect to $\rightarrow_R$ and are not E equivalent, it is the case that $(b, f(a,b)) \notin \rightarrow^*_R \circ \leftrightarrow^*_E \circ \leftarrow^*_R$. Therefore R is not locally coherent with respect to E.

R/E is Church-Rosser: This can be easily proved using Lemma 4.1. In fact, we could prove that $E \setminus R$ is Church-Rosser as follows:

[2]This example also shows that Lemma 2.2 in [13] is erroneous.

- *R/E is terminating*: Obvious, since $\rightarrow_R$ decreases size and $\leftrightarrow_E$ preserves size.
- *$E \setminus R$ is locally coherent and locally confluent*:

  Note that $CP_E(R, E)$ is empty since $f(x,x)$ cannot be E-unified with a or b. The only critical pair in $CP_E(R, R)$ is trivial since $f(x,x) \leftrightarrow_E^* f(y,y)$ iff $x \leftrightarrow_E^* y$.

□

Lemma 5.6 *It is not the case that if $R_1 \cup R_2$ is inter-reduced and $E_2 \setminus R_1$ is Church-Rosser then $R_1 \cup R_2$ is confluent.*

Proof.

The proof as in the previous cases is again by means of a counter example.

$$R_1: \quad b(a(x)) \rightarrow x$$

$$R_2: \quad a(c(x)) \rightarrow c(a(x))$$

Note that $b(a(c(x))) \rightarrow_{R_1} c(x)$ and $b(a(c(x))) \rightarrow_{R_2} b(c(a(x)))$. It is easy to see that $(c(x), b(c(a(x)))) \notin \rightarrow_{R_1 \cup R_2}^* \circ \leftarrow_{R_1 \cup R_2}^*$. Hence $R_1 \cup R_2$ is not confluent.

It is obvious that $R_1 \cup R_2$ is a (inter-)reduced rewrite system. We complete the proof by illustrating that $E_2 \setminus R_1$ is Church-Rosser, where E_2 is obtained from R_2 by treating the rules as equations, by using the E-critical pair lemma.

- *R_1/E_2 is terminating* : This is trivially true since each application of R_1/E_2 results in a strict decrease in the size of the term.
- *$E_2 \setminus R_1$ is locally confluent and locally coherent:* We use the critical pair criteria to establish this. Note that $CP_{E_2}(R_1, E_2)$ is empty since $b(a(x))$ cannot be E_2-unified with $a(c(x))$, $c(a(x))$ or any of their non-variable subterms. Furthermore, the only critical pair in $CP_{E_2}(R_1, R_1)$ is by overlap of $b(a(x))$ with itself with x renamed to y. This is trivial since $b(a(x)) \leftrightarrow_E^* b(a(y))$ iff $x \leftrightarrow_E^* y$.

□

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